ODE Lecture Notes, Part II, for Fall Semester 2024

Revised on 2024-10-19

Remark 0.1 ***This is the second part for my 2024 fall semester ODE course.***

Remark 0.2 *This note is based on the textbook* ***"Elementary Differential Equations & Boundary Value Problems, 10th Edition" by Boyce & DiPrima.*** *However,* ***I will not follow the book exactly****. Lecture notes will be given to you via email whenever necessary.*

1 Second order homogeneous linear equations with constant coefficients (this is Section 3.1 of the book; see p. 137).

We say equation $y'' = f(t, y(t), y'(t))$ is a **linear** second order ODE if $f(t, y, y')$ is **linear** in y and y' (but not linear in t), i.e., if

$$f(t, y, y') = a(t) + b(t)\,y + c(t)\,y'$$

for some functions $a(t),\ b(t),\ c(t)$. In conclusion, a linear second order ODE can be written in the standard form as

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = g(t), \quad t \in I \tag{1}$$

where $p(t),\ q(t)$ and $g(t)$ are given **continuous** functions (this is minimal requirement) defined on some common interval I.

We say equation (1) is **homogeneous** if $g(t) \equiv 0$ everywhere, i.e., we have

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0, \quad t \in I. \tag{2}$$

Otherwise we say equation (1) is **nonhomogeneous**.

Sometimes we write a linear second order ODE in the more **symmetric** form:

$$P(t)\,y''(t) + Q(t)\,y'(t) + R(t)\,y(t) = G(t), \quad t \in I \tag{3}$$

where $P(t),\ Q(t),\ R(t),\ G(t)$ are continuous on I, with $P(t) \neq 0$ on I.

Theorem 1.1 ***(Existence and uniqueness.)*** *Consider equation (3), where $P(t),\ Q(t),\ R(t),\ G(t)$ are* ***continuous*** *on I, with $P(t) \neq 0$ on I. We have the following properties:*

1. *Any solution $y(t)$ to equation (3) is defined on the* ***whole interval*** *I. Note: Since equation is of second order, by definition, $y(t)$ must be C^2 on I.*

2. *We have* ***existence and uniqueness*** *result for equation (3) with the initial condition*

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \tag{4}$$

where $t_0 \in I$ and $y_0,\ z_0$ are two given arbitrary numbers. Uniqueness means that any two solutions $y_1(t)$ and $y_2(t)$ defined on I and satisfy the same initial condition (4) must be identical on I.

Proof. We will not prove the theorem in detail. You can look for it from some ODE textbook. Roughly speaking, if we write the ODE in the form

$$y'' = f(t, y, y'),$$

then the function $f(t, u, v)$ is defined on the domain $D = I \times \mathbb{R} \times \mathbb{R} \subset \mathbb{R}^3$ and is given by

$$f(t, u, v) = \frac{G(t)}{P(t)} - \frac{Q(t)}{P(t)}v - \frac{R(t)}{P(t)}u, \quad P(t) \neq 0 \text{ on } I, \quad (t, u, v) \in D.$$

$f(t, u, v)$ is **continuous** on D and for a given initial condition (4), it is **Lipschitz continuous** with respect to (u, v) near the point $(t_0, y_0, z_0) \in D$, i.e. there exist some open set $\tilde{D} \subset D$ containing (t_0, y_0, z_0) and a constant $K > 0$ such that

$$|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq K |(u_1, v_1) - (u_2, v_2)|, \quad \forall \ (t, u_1, v_1), \ (t, u_1, v_1) \in \tilde{D} \subset D.$$

Therefore, similar to the case of a first order ODE $x' = f(t, x)$, we have **existence and uniqueness** result. □

Remark 1.2 *(**Important.**) In the above theorem, the condition $P(t) \neq 0$ on I is essential. If $P(t_0) = 0$ for some $t_0 \in I$, then a solution $y(t)$ to equation (3) **may or may not** be defined on the whole interval I. For example, the general solution of the **Euler equation** (we will learn how to solve it later on)*

$$t^2 y''(t) + t y'(t) + y(t) = 0, \quad t \in (-\infty, \infty)$$

is

$$y(t) = C_1 \cos(\log|t|) + C_2 \sin(\log|t|), \quad t \in (-\infty, 0) \bigcup (0, \infty),$$

*which **cannot be defined** across $t = 0$. On the other hand, the general solution of the equation (we will learn how to solve it later on)*

$$(t-1) y''(t) - t y'(t) + y(t) = 0, \quad t \in (-\infty, \infty), \tag{5}$$

is

$$y(t) = C_1 t + C_2 e^t, \quad t \in (-\infty, \infty). \tag{6}$$

*We see that, surprisingly, any solution $y(t)$ of (5) **can be defined** across $t = 1$.*

Another two important properties are:

Lemma 1.3 *Consider the **homogeneous** linear equation (it means that the right hand side of the equation is zero)*

$$P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = 0, \quad t \in I, \tag{7}$$

*where $P(t) \neq 0$ on I. If $y_1(t)$ and $y_2(t)$ are both solutions to (7) on I, so is the **linear combination** $c_1 y_1(t) + c_2 y_2(t)$ for any constants c_1 and c_2.*

Remark 1.4 *Hence the solution space has a **vector space structure**.*

Proof. This is a simple exercise. □

Lemma 1.5 *Consider the **nonhomogeneous** linear equation (it means that the right hand side of the equation is nonzero)*

$$P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = G(t), \quad t \in I, \tag{8}$$

where $P(t) \neq 0$ on I. If $y_1(t)$ and $y_2(t)$ are both solutions to (8) on I, then $y_2(t)$ can be expressed as

$$y_2(t) = y_h(t) + y_1(t), \quad t \in I,$$

*for some function $y_h(t)$, which is a solution of the **homogeneous** equation (7) on I. Therefore, the **general solution** $y_g(t)$ of (7) on I is given by*

$$y_g(t) = y_h(t) + y_p(t), \quad t \in I, \tag{9}$$

*where $y_p(t)$ is some **particular solution** of (8) on I ($y_p(t)$ has no integration constant) and $y_h(t)$ is the **general solution** of the **homogeneous** equation (7) on I ($y_h(t)$ has two integration constants).*

Proof. Just let $y(t) = y_2(t) - y_1(t)$ and see that $y(t)$ is a solution of the homogeneous equation. □

Solving a linear equation of the form (3) can be very difficult. Hence we first focus on the **homogeneous** case with **constant coefficients**. That is, equation of the form

$$ay''(t) + by'(t) + cy(t) = 0 \tag{10}$$

where $a,\ b,\ c$ are real constants, $a \neq 0$. By Theorem 1.1, we know that any solution $y(t)$ of (10) is defined on $(-\infty, \infty)$.

The following observation is useful: if we try $y(t) = e^{rt}$ in (10), we will get

$$ay''(t) + by'(t) + cy(t) = \left(ar^2 + br + c\right)e^{rt}. \tag{11}$$

In particular, if the number r satisfies $ar^2 + br + c = 0$, then $y(t) = e^{rt}$ will be a solution of (10).

In most cases, the equation $ar^2 + br + c = 0$ has two distinct roots $r_1,\ r_2$. By this, we can obtain the following result:

Theorem 1.6 *Assume that the polynomial equation (call it **characteristic equation** of the differential equation)*

$$ar^2 + br + c = 0, \quad a \neq 0, \quad a,\ b,\ c \in \mathbb{R} \tag{12}$$

*has **two distinct real roots** $r_1,\ r_2$. Then the **general solution** $y(t)$ to equation (10) is given by*

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad t \in (-\infty, \infty), \quad r_1 \neq r_2 \tag{13}$$

for arbitrary constants $c_1,\ c_2 \in (-\infty, \infty)$.

Proof. It is easy to check that (13) is a solution to (10) defined on $t \in (-\infty, \infty)$. On the other hand, we need to show that any solution $y(t)$ of (10) is defined on $(-\infty, \infty)$ and has the form (13) for some constants $c_1,\ c_2$. There are two ways to prove this.

Method 1: Use the existence and uniqueness theorem.

Pick a fixed $t_0 \in I$. Assume $\tilde{y}(t)$ is a solution of (10) with $\tilde{y}(t_0) = y_0$ and $\tilde{y}'(t_0) = z_0$ for some $y_0,\ z_0$. Theorem 1.1 says that it must be defined on $t \in (-\infty, \infty)$. Moreover, by linear algebra, one can find unique constants c_1 and c_2 satisfying

$$\begin{cases} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = z_0 \end{cases} \tag{14}$$

due to

$$\begin{vmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ r_1 e^{r_1 t_0} & r_2 e^{r_2 t_0} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1 + r_2) t_0} \neq 0, \quad r_1 \neq r_2.$$

Therefore, the solution $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t},\ t \in (-\infty, \infty)$, also satisfies $y(t_0) = y_0$ and $y'(t_0) = z_0$. By uniqueness property in Theorem 1.1, we must have $\tilde{y}(t) = y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ for all $t \in (-\infty, \infty)$.

Method 2: Decompose the second order ODE (10) into two first order ODE.

Remark 1.7 *This method is under the condition that the* ***characteristic equation*** *of the ODE has two* ***real*** *roots $r_1,\ r_2$. Otherwise, we may have to look at* ***complex solutions*** *of the ODE (we will avoid them in this course).*

For convenience of notation, we let D denote the differential operator d/dt. We can write $y'(t)$ as $Dy(t)$ and write $y''(t)$ as $D(Dy(t))$ or $D^2 y(t)$. Now we can write (10) as

$$ay''(t) + by'(t) + cy(t) = \left(aD^2 + bD + c\right) y(t) = 0.$$

We have:

Lemma 1.8 *(**Decomposing second order ODE into two first order ODE.**) Let $a,\ b,\ c$ be real constants with $a \neq 0$. If $r_1,\ r_2$ are two real roots (**repeated or not**) of the polynomial $aD^2 + bD + c = 0$, then we have*

$$\left(aD^2 + bD + c\right) y(t) = a(D - r_1) w(t), \tag{15}$$

where

$$w(t) = (D - r_2) y(t). \tag{16}$$

Proof. We have

$$a(D - r_1) w(t) = a(D - r_1)[(D - r_2) y(t)] = a(D - r_1) \underbrace{(y'(t) - r_2 y(t))}$$

$$= a\left[D \underbrace{(y'(t) - r_2 y(t))} - r_1 \underbrace{(y'(t) - r_2 y(t))}\right] = a\left[y''(t) - r_2 y'(t) - r_1 y'(t) + r_1 r_2 y(t)\right]$$

$$= a\left[y''(t) - (r_1 + r_2) y'(t) + r_1 r_2 y(t)\right] = ay''(t) + by'(t) + cy(t),$$

where we have used the identity $r_1 + r_2 = -b/a,\ r_1 r_2 = c/a$. □

Back to the proof of Theorem 1.6:

If $r_1 \neq r_2$ are two **real** roots of $ar^2 + br + c = 0$, then we have

$$0 = ay''(t) + by'(t) + cy(t) = a(D - r_1) w(t), \quad \text{where } w(t) = (D - r_2) y(t).$$

Hence $w(t)$ satisfies the **first order** equation $a(D - r_1) w(t) = 0$ and its general solution is given by $w(t) = \lambda e^{r_1 t}$, $\lambda \in \mathbb{R}$. But since $w(t) = (D - r_2) y(t)$, we need to solve $y(t)$ satisfying

$$(D - r_2) y(t) = \lambda e^{r_1 t} \ \ \text{(same as } y'(t) - r_2 y(t) = \lambda e^{r_1 t}\text{)}.$$

The general solution for $y(t)$ is

$$y(t) = e^{\int r_2 dt}\left\{\int \left(e^{\int -r_2 dt} \lambda e^{r_1 t}\right) dt + C\right\} = e^{r_2 t}\left\{\lambda \int e^{(r_1 - r_2)t} dt + C\right\}$$

$$= e^{r_2 t}\left\{\frac{\lambda}{r_1 - r_2} e^{(r_1 - r_2)t} + C\right\} \ (r_1 \neq r_2) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad t \in (-\infty, \infty)$$

for some constants $c_1,\ c_2$. Hence we see that any solution $y(t)$ to equation (10) is given by (13). The proof is done. □

Remark 1.9 *In case there are initial conditions for ODE (10), given by*

$$y(t_0) = y_0, \quad y'(t_0) = z_0,$$

then one can ***always*** *solve for* c_1 *and* c_2 *to fulfill them. We need to solve the system*

$$\begin{cases} c_1 e^{r_1 t_0} + c_2 e^{r_2 t_0} = y_0 \\ c_1 r_1 e^{r_1 t_0} + c_2 r_2 e^{r_2 t_0} = z_0. \end{cases}$$

Since the coefficients determinant is nonzero, given by

$$\begin{vmatrix} e^{r_1 t_0} & e^{r_2 t_0} \\ r_1 e^{r_1 t_0} & r_2 e^{r_2 t_0} \end{vmatrix} = (r_2 - r_1)\, e^{(r_1+r_2)t_0} \neq 0, \quad r_1 \neq r_2,$$

we can solve for c_1 *and* c_2 *uniquely.*

Similarly, we have:

Theorem 1.10 *Assume that the characteristic equation*

$$ar^2 + br + c = 0, \quad a \neq 0, \quad a,\ b,\ c \in \mathbb{R} \tag{17}$$

has ***two repeated real roots*** $r_1 = r_2$ *(call it* r*). Then the* ***general solution*** $y(t)$ *to equation (10) is given by*

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}, \quad t \in (-\infty, \infty) \tag{18}$$

for arbitrary constants $c_1,\ c_2 \in (-\infty, \infty)$.

Proof. We use the above Method 2 to prove it. It is easy to check that any function of the form (18) is a solution of equation (10) (you need to use the fact that $2ar + b = 0$, i.e. $r = -b/2a$). On the other hand, if $y(t)$ satisfies (10), then by

$$0 = ay''(t) + by'(t) + cy(t) = a(D - r)\, w(t), \quad \text{where } w(t) = (D - r)\, y(t),$$

we see that $w(t)$ must have the form $w(t) = \lambda e^{rt}$, $\lambda \in \mathbb{R}$. Hence $y(t)$ must satisfy

$$(D - r)\, y(t) = \lambda e^{rt}.$$

This implies

$$y(t) = e^{\int r dt}\left\{\int \left(e^{\int -r dt} \lambda e^{rt}\right) dt + C\right\} = e^{rt}(\lambda t + C) = c_1 e^{rt} + c_2 t e^{rt}, \quad t \in (-\infty, \infty)$$

for some constants $c_1,\ c_2$. The proof is done. □

Remark 1.11 *In case there are initial conditions given by*

$$y(t_0) = y_0, \quad y'(t_0) = z_0,$$

then one can ***always*** *solve for* c_1 *and* c_2 *to fulfill them. We need to solve the system*

$$\begin{cases} c_1 e^{r t_0} + c_2 t_0 e^{r t_0} = y_0 \\ c_1 r e^{r t_0} + c_2 (1 + r t_0)\, e^{r t_0} = z_0. \end{cases}$$

Since the coefficients determinant is nonzero, given by

$$\begin{vmatrix} e^{r t_0} & t_0 e^{r t_0} \\ r e^{r t_0} & (1 + r t_0)\, e^{r t_0} \end{vmatrix} = e^{2 r t_0} \neq 0 \quad \text{for any} \quad t_0,$$

we can solve for c_1 *and* c_2 *uniquely.*

To go to the last case (**two complex conjugate roots** $r=\alpha+i\beta,\ \bar{r}=\alpha-i\beta$), we first need the following:

Lemma 1.12 *The* ***general solution*** *of the equation* $(a=1,\ b=0,\ c=1)$

$$y''(t)+y(t)=0 \tag{19}$$

is given by

$$y(t)=c_1\cos t+c_2\sin t,\quad t\in(-\infty,\infty), \tag{20}$$

for arbitrary constants $c_1,\ c_2\in(-\infty,\infty)$.

Proof. We already know that, for arbitrary constants $c_1,\ c_2$, (20) is a solution of equation (10). It remains to claim that any solution of (19) on $(-\infty,\infty)$ has the form (20). Let $Y(t)$ be any solution of (19) on $(-\infty,\infty)$ with $Y(0)=a,\ Y'(0)=b$. We can choose $c_1=a,\ c_2=b$ in (20) and the solution $y(t)=a\cos t+b\sin t$ also satisfies $y(0)=a,\ y'(0)=b$. By uniqueness property in Theorem 1.1, we must have $Y(t)=y(t)$ for all $t\in(-\infty,\infty)$. Hence the general solution of (19) is given by $y(t)$ in (20). The proof is done. □

Now we come to the last case for the equation (10):

Theorem 1.13 *Assume that the characteristic equation*

$$ar^2+br+c=0,\quad a\neq 0,\quad a,\ b,\ c\in\mathbb{R} \tag{21}$$

has ***two complex conjugate roots*** $r=\alpha+i\beta,\ \bar{r}=\alpha-i\beta,\ \alpha,\ \beta\in\mathbb{R},\ \beta>0$. *Then the* ***general (real) solution*** $y(t)$ *to equation (10) is given by*

$$y(t)=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t,\quad t\in(-\infty,\infty) \tag{22}$$

for arbitrary constants $c_1,\ c_2\in(-\infty,\infty)$.

To prove the above theorem, we first note the following "change of variables" trick:

Lemma 1.14 *Consider the equation*

$$ay''(t)+by'(t)+cy(t)=0 \tag{23}$$

where $a,\ b,\ c$ *are real constants,* $a\neq 0$ *(we assume* $a>0$*). Then one can find a function of the form* $v(t)=e^{\lambda t}y(t)$ *for some constant* $\lambda\in\mathbb{R}$ *so that the equation for* $v(t)$ *is given by*

$$av''(t)+Bv(t)=0 \tag{24}$$

for some number B. *More precisely, if we choose* $\lambda=\frac{b}{2a}$ *in* $v(t)=e^{\lambda t}y(t)$, *then equation (23) becomes*

$$av''(t)+\left(\frac{4ac-b^2}{4a}\right)v(t)=0,\quad v(t)=e^{\left(\frac{b}{2a}\right)t}y(t) \tag{25}$$

and if the characteristic equation $ar^2+br+c=0$ *of the ODE has* ***two complex conjugate roots***

$$r=\alpha+i\beta,\quad \bar{r}=\alpha-i\beta,\quad \alpha,\ \beta\in\mathbb{R},\quad \beta>0, \tag{26}$$

then

$$\lambda=-\alpha,\qquad B=a\beta^2. \tag{27}$$

In such a case, equation (25) becomes (after cancelling a*)*

$$v''(t)+\beta^2v(t)=0,\quad \text{where now}\quad v(t)=e^{-\alpha t}y(t). \tag{28}$$

Proof. Let $v(t) = e^{\lambda t} y(t)$, where the constant λ is to be chosen later on, we have $y(t) = e^{-\lambda t} v(t)$ and then

$$\begin{aligned} 0 &= ay''(t) + by'(t) + cy(t) \\ &= a\left[\lambda^2 e^{-\lambda t} v(t) - 2\lambda e^{-\lambda t} v'(t) + e^{-\lambda t} v''(t)\right] + b\left[-\lambda e^{-\lambda t} v(t) + e^{-\lambda t} v'(t)\right] + ce^{-\lambda t} v(t). \end{aligned}$$

Therefore, we have

$$a\left[\lambda^2 v(t) - 2\lambda v'(t) + v''(t)\right] + b\left[-\lambda v(t) + v'(t)\right] + cv(t) = 0,$$

which is the same as

$$av''(t) + \underbrace{(b - 2\lambda a)}\, v'(t) + \left(\lambda^2 a - \lambda b + c\right) v(t) = 0. \tag{29}$$

If we choose $\lambda = b/(2a)$, we get

$$b - 2\lambda a = 0, \qquad \left(\frac{b}{2a}\right)^2 a - \left(\frac{b}{2a}\right) b + c = \frac{4ac - b^2}{4a}$$

and the equation (29) for $v(t)$ becomes

$$av''(t) + \left(\frac{4ac - b^2}{4a}\right) v(t) = 0, \tag{30}$$

which gives the equation (25).

If the two characteristic roots are given by (26), we have

$$\alpha \pm i\beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \; (b^2 - 4ac < 0), \quad \alpha = \frac{-b}{2a}, \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} > 0,$$

and

$$\lambda = \frac{b}{2a} = -\alpha, \quad B = \frac{4ac - b^2}{4a} = a\beta^2. \tag{31}$$

Under (31) the equation (28) follows. The proof is done. □

Lemma 1.15 *Consider the equation*

$$v''(t) + \beta^2 v(t) = 0, \quad \beta > 0 \text{ is a constant.} \tag{32}$$

Its given by is given by

$$v(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t), \quad t \in (-\infty, \infty), \tag{33}$$

for arbitrary constants $c_1, \; c_2 \in (-\infty, \infty)$.

Proof. Again we can use a "change of variables" trick as before. Let $Y(s)$ be the function satisfying $Y(\beta t) = v(t)$ (same as $Y(s) = v\left(\frac{s}{\beta}\right)$). By the chain rule we have

$$v'(t) = \beta Y'(s), \quad v''(t) = \beta^2 Y''(s), \quad \text{where } s = \beta t$$

and, in terms of $Y(s)$, the equation (32) becomes

$$\beta^2 Y''(\beta t) + \beta^2 Y(\beta t) = 0 \text{ (same as } Y''(s) + Y(s) = 0)$$

and by Lemma 1.12 we know that the general solution for $Y(s)$ is given by

$$Y(s) = c_1 \cos s + c_2 \sin s, \quad s \in (-\infty, \infty)$$

for arbitrary constants $c_1,\ c_2 \in (-\infty, \infty)$. Hence the general solution for $v(t)$ is given by (note that the solutions of $Y(s)$ are in one-one correspondence to $v(t)$)

$$v(t) = Y(s) = c_1 \cos s + c_2 \sin s = c_1 \cos(\beta t) + c_2 \sin(\beta t).$$

The proof is done. □

Now we are ready to prove Theorem 1.13:

Proof. Consider the equation (10) with $r = \alpha + i\beta$ and $\bar{r} = \alpha - i\beta,\ \beta > 0$, as its characteristic polynomial. By Lemma 1.14 and (27), the function $v(t) = e^{-\alpha t} y(t)$ is given by

$$v(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t).$$

Therefore, the general solution for $y(t)$ is given by

$$y(t) = e^{\alpha t} v(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t), \quad t \in (-\infty, \infty). \tag{34}$$

The proof of Theorem 1.13 is done. □

Remark 1.16 *Again in case there is initial conditions for ODE (10), given by*

$$y(t_0) = y_0, \quad y'(t_0) = z_0$$

then one can ***always*** *solve for c_1 and c_2 to fulfill them. We need to solve*

$$\begin{cases} c_1 e^{\alpha t_0} \cos \beta t_0 + c_2 e^{\alpha t_0} \sin \beta t_0 = y_0 \\ c_1 \left(\alpha e^{\alpha t_0} \cos \beta t_0 - \beta e^{\alpha t_0} \sin \beta t_0\right) + c_2 \left(\alpha e^{\alpha t_0} \sin \beta t_0 + \beta e^{\alpha t_0} \cos \beta t_0\right) = y_0' \end{cases}$$

and the coefficients determinant is nonzero, given by

$$\begin{vmatrix} e^{\alpha t_0} \cos \beta t_0 & e^{\alpha t_0} \sin \beta t_0 \\ \alpha e^{\alpha t_0} \cos \beta t_0 - \beta e^{\alpha t_0} \sin \beta t_0 & \alpha e^{\alpha t_0} \sin \beta t_0 + \beta e^{\alpha t_0} \cos \beta t_0 \end{vmatrix} = \beta e^{2\alpha t_0} \neq 0 \quad \text{for any} \quad t_0,$$

due to $\beta > 0$.

At this moment, we can summarize the following:

Theorem 1.17 *The ODE with initial conditions:*

$$\begin{cases} a y''(t) + b y'(t) + c y(t) = 0 \\ y(t_0) = y_0, \quad y'(t_0) = z_0, \end{cases} \tag{35}$$

where $a,\ b,\ c$ are real constants with $a \neq 0$, has a ***unique real*** *solution $y(t)$ defined on $(-\infty, \infty)$. Moreover, we know how to find the general solution explicitly.*

1.1 Euler equations (this is Exercise 34 in p. 166).

In this section, we consider a second order **linear** equation (with **variable coefficients**) of the form (call it **Euler equation**)

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = 0, \quad t \in (0, \infty), \quad \alpha, \ \beta \text{ constants.} \tag{36}$$

One can use **change of variables** to convert it into a **linear equation with constant coefficients**. Let $x = \ln t, \ t \in (0, \infty)$, (same as $t = e^x$), where $x \in (-\infty, \infty)$ will be the new variable. The function $y(t)$ will become a function of x, which we denote it as $\tilde{y}(x)$. That is, $\tilde{y}(x) = y(t)$. We have

$$\frac{dy}{dt} = \frac{d\tilde{y}}{dx}\frac{dx}{dt} = \frac{d\tilde{y}}{dx}\frac{1}{t} = e^{-x}\frac{d\tilde{y}}{dx}, \tag{37}$$

which is same as the following operator relation

$$\underbrace{\frac{d}{dt} = e^{-x}\frac{d}{dx}} \quad \text{or} \quad \underbrace{\frac{d}{dx} = t\frac{d}{dt}}$$

and

$$\frac{d^2 y}{dt^2} = e^{-x}\frac{d}{dx}\left(e^{-x}\frac{d\tilde{y}}{dx}\right) = -e^{-2x}\frac{d\tilde{y}}{dx} + e^{-2x}\frac{d^2\tilde{y}}{dx^2}. \tag{38}$$

Hence we obtain

$$\begin{aligned} & t^2 y''(t) + \alpha t y'(t) + \beta y(t), \quad \text{where } t = e^x \\ &= e^{2x}\left(-e^{-2x}\frac{d\tilde{y}}{dx} + e^{-2x}\frac{d^2\tilde{y}}{dx^2}\right) + \alpha e^x\left(e^{-x}\frac{d\tilde{y}}{dx}\right) + \beta\tilde{y}(x) \\ &= \frac{d^2\tilde{y}}{dx^2} + (\alpha - 1)\frac{d\tilde{y}}{dx} + \beta\tilde{y}(x), \quad x = \ln t \in (-\infty, \infty), \quad t \in (0, \infty) \end{aligned}$$

and, in terms of the new variable x, the function $\tilde{y}(x)$ will satisfy the linear equation with constant coefficients:

$$\frac{d^2\tilde{y}}{dx^2} + (\alpha - 1)\frac{d\tilde{y}}{dx} + \beta\tilde{y}(x) = 0. \tag{39}$$

One can solve $\tilde{y}(x)$ for equation (39) on the interval $x \in (-\infty, \infty)$ and then obtain solution $y(t)$ for equation (36) on the interval $t \in (0, \infty)$.

Remark 1.18 *In case the Euler equation has the form*

$$At^2 y''(t) + Bty'(t) + Cy(t) = 0, \quad t \in (0, \infty), \quad A \neq 0, \ B, \ C \text{ are constants,}$$

then equation (39) becomes

$$A\frac{d^2\tilde{y}}{dx^2} + (B - A)\frac{d\tilde{y}}{dx} + C\tilde{y}(x) = 0. \tag{40}$$

Remark 1.19 *(**Important.**) In case we focus on the Euler equation on the domain $t \in (-\infty, 0)$, then on $t \in (-\infty, 0)$ we do the change of variables:*

$$x = \log(-t), \quad t = -e^x, \quad t \in (-\infty, 0), \quad x \in (-\infty, \infty),$$

and get (still let $\tilde{y}(x) = y(t)$)

$$\frac{dy}{dt} = \frac{d\tilde{y}}{dx}\frac{dx}{dt} = \frac{d\tilde{y}}{dx}\frac{1}{t} = -e^{-x}\frac{d\tilde{y}}{dx}$$

and

$$\frac{d^2y}{dt^2} = -e^{-x}\frac{d}{dx}\left(-e^{-x}\frac{d\tilde{y}}{dx}\right) = e^{-x}\frac{d}{dx}\left(e^{-x}\frac{d\tilde{y}}{dx}\right)$$
$$= -e^{-2x}\frac{d\tilde{y}}{dx} + e^{-2x}\frac{d^2\tilde{y}}{dx^2} \text{ (same as the case } t > 0\text{)}.$$

Therefore

$$\begin{aligned} & t^2y''(t) + \alpha t y'(t) + \beta y(t), \quad \text{where now } t = -e^x \in (-\infty, 0) \\ &= e^{2x}\left(-e^{-2x}\frac{d\tilde{y}}{dx} + e^{-2x}\frac{d^2\tilde{y}}{dx^2}\right) + \alpha(-e^x)\left(-e^{-x}\frac{d\tilde{y}}{dx}\right) + \beta\tilde{y}(x) \\ &= \underbrace{\frac{d^2\tilde{y}}{dx^2} + (\alpha - 1)\frac{d\tilde{y}}{dx} + \beta\tilde{y}(x)}, \quad x = \log(-t) \in (-\infty, \infty), \quad t \in (-\infty, 0), \end{aligned}$$

and we have arrived at the ***same*** *equation as in the case* $t \in (0, \infty)$.

We can summarize the above as the following:

Theorem 1.20 *(**General solution of Euler equation**.) Consider Euler equation of the form (**be aware that the coefficient in front of** $t^2y''(t)$ **is** 1)*

$$t^2y''(t) + \alpha t y'(t) + \beta y(t) = 0, \quad t \in (0, \infty), \quad \alpha,\ \beta \text{ constants,} \tag{41}$$

where we focus on $t \in (0, \infty)$. *Let* $r_1,\ r_2$ *be the two roots of the* ***characteristic equation***

$$r^2 + (\alpha - 1)r + \beta = 0 \tag{42}$$

of the equation (39) for $\tilde{y}(x)$. *Then the general solution of (41) is given by*

$$y(t) = \begin{cases} c_1t^{r_1} + c_2t^{r_2}, & r_1 \neq r_2 \in \mathbb{R}, \\ c_1t^r + c_2t^r\log t, & r_1 = r_2 = r \in \mathbb{R}, \\ c_1t^p\cos(q\log t) + c_2t^p\sin(q\log t), & r_1 = p + iq, \quad r_2 = p - iq, \quad p,\ q \in \mathbb{R}, \quad q > 0, \end{cases} \tag{43}$$

where $t \in (0, \infty)$ *and* $c_1,\ c_2$ *are two arbitrary real constants. Also note that we have*

$$r_i = \frac{-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 4\beta}}{2}, \quad i = 1,\ 2. \tag{44}$$

Remark 1.21 *We call the polynomial equation (42)* ***the characteristic equation of the Euler equation*** *(41).*

Remark 1.22 *By Remark 1.19, in case we focus on* $t \in (-\infty, 0) \bigcup (0, \infty)$ *for the Euler equation (41), then the solution formula (43) becomes*

$$y(t) = \begin{cases} c_1|t|^{r_1} + c_2|t|^{r_2}, & r_1 \neq r_2 \in \mathbb{R}, \\ c_1|t|^r + c_2|t|^r\log|t|, & r_1 = r_2 = r \in \mathbb{R}, \\ c_1|t|^p\cos(q\log|t|) + c_2|t|^p\sin(q\log|t|), & r_1 = p + iq, \quad r_2 = p - iq, \quad p,\ q \in \mathbb{R}, \quad q > 0. \end{cases} \tag{45}$$

Example 1.23 *(This is problem 35, p. 166.) Find the general solution of the Euler equation*

$$t^2y''(t) + ty'(t) + 4y(t) = 0, \quad t \in (0, \infty). \tag{46}$$

Solution:

By the change of variables $x = \log t,\ t \in (0, \infty)$, the new equation for $\tilde{y}(x)$ is

$$\tilde{y}''(x) + 4\tilde{y}(x) = 0,$$

which has general solution

$$\tilde{y}(x) = c_1 \cos(2x) + c_2 \sin(2x), \quad x \in (-\infty, \infty).$$

Back to $y(t)$, we get

$$y(t) = c_1 \cos(2 \log t) + c_2 \sin(2 \log t), \quad t \in (0, \infty),$$

which is the general solution of the Euler equation. Here c_1 and c_2 are arbitrary constants. □

2 Theory of second order linear homogeneous equation with variable coefficients; the Wronskian (this is Section 3.2 of the book).

Remark 2.1 ***(Be careful.)*** *Throughout this section, we will focus on equation (48), which has leading coefficient* 1 *for* $y''(t)$.

In this section we look at a second order linear homogeneous equation with **variable coefficients** of the form

$$P(t)\, y''(t) + Q(t)\, y'(t) + R(t)\, y(t) = 0, \quad t \in I \tag{47}$$

where $P(t),\ Q(t),\ R(t)$ are real-valued continuous functions on I, with $P(t) \neq 0$ on I. For convenience, we can divide the equation by $P(t)$ and it becomes

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0, \quad t \in I, \tag{48}$$

where $p(t),\ q(t)$ are continuous on I. By Theorem 1.1, we know that if we have initial conditions $y(t_0) = y_0,\ y'(t_0) = z_0$, for equation (48), where $t_0 \in I$ and $y_0,\ z_0$ are two given numbers, then the solution $y(t)$ exists, unique, and is defined on the whole interval I.

Unlike the case of constant coefficients, it is, in general, very difficult to solve equation (48). However, we can ask the following interesting question: if $y_1(t)$ and $y_2(t)$ are two known solutions of (48) on I, **is it true that any other solution** $y(t)$ **of (48) can be expressed as**

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad \forall\, t \in I, \tag{49}$$

for some constants c_1 **and** c_2**?** If yes, then the **general solution** of (48) on I is given by (49).

To answer the above question, we need the concept of **Wronskian**, defined by the following:

Definition 2.2 *Consider the ODE (**homogeneous with leading coefficient*** 1*)*

$$y'' + p(t)\, y' + q(t)\, y = 0, \quad t \in I, \tag{50}$$

where $p(t),\ q(t)$ *are continuous functions defined on open interval* I. *If* $y_1(t)$ *and* $y_2(t)$ *are two solutions on* I, *then the function*

$$W(y_1, y_2)(t) := \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t), \quad t \in I,$$

*is called the **Wronskian** of* $y_1(t)$ *and* $y_2(t)$ *on* I. *We usually denote* $W(y_1, y_2)(t)$ *as* $W(t)$ *if there is no confusion appeared.*

Remark 2.3 ***In the above definition, the coefficient of*** $y''(t)$ ***is*** 1. *From now on, when we talk about the Wronskian* $W(y_1, y_2)(t)$ *of two functions* $y_1(t)$ *and* $y_2(t)$ *on* I, *we* ***always assume*** *that* $y_1(t)$ *and* $y_2(t)$ *are two* ***solutions*** *of the ODE (50) on* I *unless otherwise stated.*

The following theorem implies that the Wronskian $W(y_1, y_2)(t)$ of two solutions on I is **either everywhere zero or everywhere nonzero** on I. More precisely, we have:

Theorem 2.4 ***(Abel's formula for homogeneous equation.)*** ***(This is Theorem 3.2.7 in p. 154.)*** *Consider the ODE (50) on* I. *If* $y_1(t)$ *and* $y_2(t)$ *are two solutions on* I, *then we have*

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt}, \quad t \in I \tag{51}$$

for some constant $c \in (-\infty, \infty)$.

Remark 2.5 *Note that the ODE in (50) has* ***leading coefficient*** 1 *and is* ***homogeneous***.

Proof. For convenience, denote $W(y_1, y_2)(t)$ as $W(t)$. Compute

$$\begin{aligned} W'(t) &= y_1(t)\, y_2''(t) - y_2(t)\, y_1''(t) \\ &= y_1(t)\left[-p(t)\, y_2'(t) - q(t)\, y_2(t)\right] - y_2(t)\left[-p(t)\, y_1'(t) - q(t)\, y_1(t)\right] \\ &= -p(t)\, y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) = -p(t)\, W(t), \quad t \in I. \end{aligned}$$

Thus $W(t)$ satisfies the first order linear ODE

$$W'(t) + p(t)\, W(t) = 0, \quad t \in I.$$

Hence it is given by

$$W(t) = ce^{-\int p(t)dt}, \quad t \in I,$$

for some constant $c \in (-\infty, \infty)$. □

Remark 2.6 *If there is an initial condition* $W(y_1, y_2)(t_0) = c_0$, *then one can express (51) as*

$$W(y_1, y_2)(t) = c_0 e^{-\int_{t_0}^{t} p(s)ds}. \tag{52}$$

It satisfies

$$W'(t) + p(t)\, W(t) = 0, \quad W(t_0) = c_0, \quad t_0,\ t \in I.$$

Corollary 2.7 *By (51), we see that* $W(t)$ *is either* $W(t) \equiv 0$ *on* I *(if* $c = 0$*) or* $W(t)$ *is never zero on* I *(if* $c \neq 0$*). In particular, if* $W(t_0) = 0$ *at some* $t_0 \in I$, *then* $W(t) \equiv 0$ *everywhere on* I. *If* $W(t_0) > 0$ *at some* $t_0 \in I$, *then* $W(t) > 0$ *everywhere on* I. *If* $W(t_0) < 0$ *at some* $t_0 \in I$, *then* $W(t) < 0$ *everywhere on* I. *Here* I *is the domain interval of* $p(t)$ *and* $q(t)$ *in (50).*

Proof. This is a direct consequence of Theorem 2.4. □

Lemma 2.8 ***(The case when*** $W(t) \equiv 0$ ***everywhere.)*** *Consider the ODE (50) on* I *and* $y_1(t)$ *and* $y_2(t)$ *are two* ***nonzero*** *solutions on* I *with* $W(t_0) = 0$ *at some* $t_0 \in I$, *then we have* $W(t) \equiv 0$ *for all* $t \in I$ *and there exists a constant* $\lambda \neq 0$ *such that* $y_2(t) = \lambda y_1(t)$ *for all* $t \in I$.

Proof. We have

$$W(t_0) = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0, \quad t_0 \in I,$$

which implies $W(t) \equiv 0$ for all $t \in I$ and the two vectors $v_1 := (y_1(t_0), y_1'(t_0))$ and $v_2 := (y_2(t_0), y_2'(t_0))$ are **dependent** in $\mathbb{R}^2$. Since $y_1(t)$ and $y_2(t)$ are both **nonzero** solutions on I, we must have $v_1 \neq (0,0)$ and $v_2 \neq (0,0)$ (otherwise we get a contradiction). By linear algebra, we know that $v_2 = \lambda v_1$ for some constant $\lambda \neq 0$, which gives

$$y_2(t_0) - \lambda y_1(t_0) = 0, \quad y_2'(t_0) - \lambda y_1'(t_0) = 0.$$

Since $y_2(t) - \lambda y_1(t)$ is also a solution of the ODE (50) on I, by uniqueness theorem, we must have $y_2(t) - \lambda y_1(t) \equiv 0$ for all $t \in I$. The proof is done. □

Remark 2.9 ***(Interesting property.)*** *By Lemma 2.8 and its proof we see that if $y_1(t)$ and $y_2(t)$ are two solutions on I with $W(t_0) > 0$ at some $t_0 \in I$, then we have $W(t) > 0$ for all $t \in I$. In particular, both are* ***nonzero solutions*** *on I and the two vectors*

$$v_1(t) = \begin{pmatrix} y_1(t) \\ y_1'(t) \end{pmatrix}, \quad v_2(t) = \begin{pmatrix} y_2(t) \\ y_2'(t) \end{pmatrix}, \quad t \in I \tag{53}$$

form a ***basis*** *in $\mathbb{R}^2$ for all $t \in I$. The same conclusion holds for $W(t_0) < 0$ at some $t_0 \in I$.*

The most important theorem in this section is the following:

Theorem 2.10 ***(This is Theorem 3.2.4 in p. 149.)*** *Consider the ODE (homogeneous)*

$$y'' + p(t)\,y' + q(t)\,y = 0 \tag{54}$$

where $p(t)$, $q(t)$ are continuous functions defined on open interval I. If $y_1(t)$ and $y_2(t)$ are two solutions on I, then the family of solutions

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad t \in I, \tag{55}$$

with ***arbitrary*** *coefficients c_1, c_2, will* ***generate all possible solutions*** *of (54) on I* ***if and only if*** *$W(y_1, y_2)(t_0) \neq 0$ at some $t_0 \in I$ (same as $W(y_1, y_2)(t) \neq 0$ for all $t \in I$).*

Proof. Assume $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Then Remark 2.9 implies that both $y_1(t)$ and $y_2(t)$ are nonzero solutions and the two vectors $\{v_1(t), v_2(t)\}$ in (53) is a basis in $\mathbb{R}^2$ for all $t \in I$. Let $\varphi(t)$ be a solution of (54) on I with

$$\varphi(t_0) = a, \quad \varphi'(t_0) = b.$$

One can find unique constants c_1 and c_2 such that

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = a \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = b \end{cases} \quad \text{(same as } c_1 v_1(t_0) + c_2 v_2(t_0) = \begin{pmatrix} a \\ b \end{pmatrix}\text{)}.$$

Now the linear combination $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution of the ODE on I with

$$y(t_0) = a, \quad y'(t_0) = b.$$

Uniqueness implies that $y(t) \equiv \varphi(t)$ for all $t \in I$.

Conversely, assume that **every solution** $\varphi(t)$ of (54) on I can be expressed in the form $c_1 y_1(t) + c_2 y_2(t)$. Fix a time $t_0 \in I$, for any solution $\varphi(t)$ on I we have the vector identity

$$\begin{pmatrix} \varphi(t_0) \\ \varphi'(t_0) \end{pmatrix} = c_1 \begin{pmatrix} y_1(t_0) \\ y_1'(t_0) \end{pmatrix} + c_2 \begin{pmatrix} y_2(t_0) \\ y_2'(t_0) \end{pmatrix} = c_1 v_1(t_0) + c_2 v_2(t_0) \tag{56}$$

for some constants c_1 and c_2. Since the vector $v(t_0) = (\varphi(t_0), \varphi'(t_0))$ can be **arbitrary** (due to the **existence** of a solution $\varphi(t)$ with $\varphi(t_0) = a$ and $\varphi'(t_0) = b$ for any a, b), the two vectors $\{v_1(t_0), v_2(t_0)\}$ in (56) must be a **basis** in $\mathbb{R}^2$ and we have

$$W(t_0) = W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0,$$

which implies $W(y_1, y_2)(t) \neq 0$ for all $t \in I$. □

By the above theorem, we define the following:

Definition 2.11 *Consider the ODE*

$$y'' + p(t)\, y' + q(t)\, y = 0, \quad p(t),\ q(t) \ \textit{continuous on } I. \tag{57}$$

If $y_1(t)$ and $y_2(t)$ are two solutions on I such that $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$ (same as $W(y_1, y_2)(t) \neq 0$ for all $t \in I$), then the ***general solution of (57) on*** *I is given by*

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad t \in I,$$

where $c_1,\ c_2$ are arbitrary constants. In this case, we call $\{y_1(t),\ y_2(t)\}$ a ***fundamental set of solutions*** *of ODE (57) on I. Note that a given ODE on I has infinitely many* ***fundamental set of solutions*** *on I.*

Remark 2.12 *If $\{y_1(t),\ y_2(t)\}$ is a* ***fundamental set of solutions****, the two vectors in (53) form a* ***basis*** *in $\mathbb{R}^2$ for all $t \in I$.*

Example 2.13 *Given the linear equation*

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0, \quad t \in (-\infty, \infty) \tag{58}$$

where $p(t)$ and $q(t)$ are continuous on the interval $(-\infty, \infty)$. Is it possible that both t and t^2 are solutions (for ***some*** *continuous $p(t)$ and $q(t)$ on $(-\infty, \infty)$) to the equation on $(-\infty, \infty)$? Give your reasons.*

<u>**Solution**</u>:

It is impossible. The Wronskian of t and t^2 is

$$W(t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = t^2, \quad t \in (-\infty, \infty).$$

On the interval $I = (-\infty, \infty)$ we have $W(t) = 0$ at $t = 0$, but $W(t) \neq 0$ for $t \neq 0$. It violates Theorem 2.4 □

Example 2.14 *Find an ODE of the form*

$$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0, \quad t \in (0, \infty), \tag{59}$$

where $p(t)$ and $q(t)$ are continuous functions on the interval $(0, \infty)$ so that both t and t^2 are solutions to the equation on $(0, \infty)$.

<u>**Solution**</u>:

Over the interval $(0, \infty)$, we have Wronskian $W(t) = t^2 \neq 0$ on $(0, \infty)$, which will not give us any contradiction so far. Hence we guess it is likely to find some $p(t)$ and $q(t)$ on $(0, \infty)$ so that t and t^2 are solutions to (59) on $(0, \infty)$.

If we want t and t^2 to be solutions of (59) on $t \in (0, \infty)$, we must require

$$\begin{cases} p(t) + q(t)\, t = 0 \\ 2 + 2tp(t) + q(t)\, t^2 = 0, \quad \forall\, t \in (0, \infty), \end{cases}$$

which gives $p(t) = -q(t)\, t$ for all $t \in (0, \infty)$ and then

$$2 + 2tp(t) + q(t)\, t^2 = 2 + 2t(-q(t)\, t) + q(t)\, t^2 = 2 - q(t)\, t^2 = 0, \quad \forall\, t \in (0, \infty),$$

and we conclude $q(t) = 2/t^2$ and $p(t) = -2/t$. Both $p(t)$ and $q(t)$ are continuous on the interval $(0,\infty)$. Therefore, the equation

$$y''(t) - \frac{2}{t}y'(t) + \frac{2}{t^2}y(t) = 0, \quad t \in (0,\infty) \tag{60}$$

has both t and t^2 as solutions on the interval $(0,\infty)$. By Theorem 2.10, they form a **fundamental set of solutions** on $(0,\infty)$. The **general solution** of the ODE (60) is given by

$$y(t) = c_1 t + c_2 t^2, \quad t \in (0,\infty)$$

for arbitrary real constants c_1 and c_2. □

Remark 2.15 *Note that (60) is an Euler equation on $(0,\infty)$ with characteristic equation*

$$r^2 - 3r + 2 = 0.$$

Example 2.16 *Consider the equation*

$$y'' - 3y' + 2y = 0.$$

We know that $y_1(t) = e^t$ and $y_2(t) = e^{2t}$ are two solutions of it defined on $t \in (-\infty,\infty)$. By

$$W(y_1,y_2)(t) = e^{3t} > 0 \quad \text{for all} \quad t \in (-\infty,\infty),$$

*we know that they form a **fundamental set of solutions** and every solution $y(t)$ of the equation has the form*

$$y(t) = c_1 e^t + c_2 e^{2t}, \quad t \in (-\infty,\infty).$$

*We also knew this fact before by **decomposing the second order ODE into two first order ODE.***

Example 2.17 *Consider the equation*

$$y'' + p(t)y' + q(t)y = 0, \quad p(t),\ q(t) \ \text{continuous on } I$$

and fix some $t_0 \in I$. Let $y_1(t)$ be the solution satisfying

$$y_1(t_0) = 1, \quad y_1'(t_0) = 0$$

and let $y_2(t)$ be the solution satisfying

$$y_2(t_0) = 0, \quad y_2'(t_0) = 1.$$

*Then $y_1(t)$ and $y_2(t)$ form a **fundamental set of solutions** (since $W(y_1,y_2)(t_0) = 1 \neq 0$).*

Example 2.18 *Do Example 6 in p. 152.*

In general, it is very difficult to solve an equation of the form $y'' + p(t)y' + q(t)y = 0$ (or the form $P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = 0$) for arbitrary continuous functions $p(t)$ and $q(t)$ on I. However, there is a special case we can solve it, which is known as **Euler equation**.

2.1 The method of "reduction of order" (this is part of Section 3.4; see p. 171).

Remark 2.19 ***(Be careful.)*** *Throughout this section, we will focus on equation (61) below, which has **leading coefficient** 1 for $y''(t)$.*

2.1.1 Reduction method for homogeneous equations.

Consider the second order linear homogeneous equation

$$y''(t) + p(t)\,y'(t) + q(t)\,y(t) = 0, \quad t \in I \tag{61}$$

and assume that **we already know one nonzero solution** $y_1(t)$ defined on $t \in I$ and assume $y_1(t) \neq 0$ on I (if $y_1(t_0) = 0$ at some $t_0 \in I$, we focus on subinterval $J \subset I$ such that $y_1(t) \neq 0$ on J). We can use the following "**reduction method**" to find another solution $y_2(t)$.

Let

$$y_2(t) = v(t)\,y_1(t), \quad t \in I.$$

The idea is to find suitable $v(t)$ so that the above $y_2(t)$ will be a solution of (61) different from $y_1(t)$. We substitute $y_2(t)$ into the equation to get

$$\begin{aligned}
0 &= y_2''(t) + p(t)\,y_2'(t) + q(t)\,y_2(t) \\
&= y_1(t)\,v''(t) + [2y_1'(t) + p(t)\,y_1(t)]\,v'(t) + \left[\underbrace{y_1''(t) + p(t)\,y_1'(t) + q(t)\,y_1(t)}\right] v(t), \quad \left[\underbrace{\cdots}\right] = 0 \\
&= y_1(t)\,v''(t) + [2y_1'(t) + p(t)\,y_1(t)]\,v'(t) \\
&= y_1(t)\,w'(t) + [2y_1'(t) + p(t)\,y_1(t)]\,w(t), \quad \text{where} \quad w(t) := v'(t), \quad t \in I.
\end{aligned} \tag{62}$$

We see that equation (62) is a **first order linear homogeneous equation** for $w(t)$ and since $y_1(t) \neq 0$ on I, any solution $w(t)$ of (62) is **defined on** I and if $w(t) = v'(t)$ is a **nonzero solution** on I, then $v(t) = \int w(t)\,dt$ will be a **non-constant solution** on I and $y_2(t) = v(t)\,y_1(t)$ will be a new solution of (61) different from $y_1(t)$ on I. After that, one can find a different solution $y_2(t)$. Finally we check the **Wronskian** $W(y_1, y_2)(t)$ and if there is one point $t_0 \in I$ such that $W(y_1, y_2)(t_0) \neq 0$, then every solution $y(t)$ to the linear equation (61) is of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1,\ c_2 \text{ are arbitrary constants.}$$

Remark 2.20 ***(Be careful.)*** *In case equation (61) has the form*

$$P(t)\,y''(t) + Q(t)\,y'(t) + q(t)\,y(t) = 0, \quad t \in I, \quad P(t) \neq 0 \text{ on } I, \tag{63}$$

then you can either rewrite it as the form (61) and use the equation (62) for $v(t)$ or you can maintain the original equation (63) and now equation (62) becomes

$$P(t)\,y_1(t)\,v''(t) + [2P(t)\,y_1'(t) + Q(t)\,y_1(t)]\,v'(t) = 0. \tag{64}$$

You can use either way. However, I would suggest you to ***rewrite equation as the form (61) first*** *because when you solve the first order equation for $w = v'$ in (64), you still have to divide the equation by $P(t)\,y_1(t)$ in order to find the* ***integrating factor****.*

Example 2.21 *Use* ***method of reduction*** *to find the general solution of*

$$y''(t) + 4y'(t) + 4y(t) = 0, \quad t \in (-\infty, \infty).$$

<u>**Solution**</u>:

We first know that $y_1(t) = e^{-2t}$ is one solution. To find the second solution, let

$$y_2(t) = v(t)\,e^{-2t}$$

and plug it into the equation to get $y_1(t)\,v''(t) + [2y_1'(t) + p(t)\,y_1(t)]\,v'(t) = 0$, i.e.

$$e^{-2t}v''(t) = 0. \tag{65}$$

Thus

$$v(t) = at + b, \quad a, \ b \text{ are arbitrary const.}$$

and $y_2(t) = (at+b)e^{-2t}$, which gives a new solution te^{-2t}. Therefore, the general solution is given by (since e^{-2t} and te^{-2t} have nonzero Wronskian)

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}, \quad t \in (-\infty, \infty),$$

which is the same as before. □

Example 2.22 *One can easily see that $y_1(t) = t$ is a solution of the following ODE on $t \in (1, \infty)$. Use **method of reduction** to find the general solution of the equation*

$$(t-1)y''(t) - ty'(t) + y(t) = 0, \quad t \in (1, \infty). \tag{66}$$

Solution:

We first rewrite the equation as the form (61) (in order to have leading coefficient 1):

$$y''(t) - \frac{t}{t-1}y'(t) + \frac{1}{t-1}y(t) = 0, \quad t \in (1, \infty).$$

Let $y_2(t) = y_1(t)v(t) = tv(t)$. We will have

$$y_1(t)v''(t) + [2y_1'(t) + p(t)y_1(t)]v'(t) = 0, \quad y_1(t) = t,$$

which is

$$tv''(t) + \left(2 - \frac{t^2}{t-1}\right)v'(t) = 0,$$

i.e.

$$w'(t) + \left(\frac{2}{t} - \frac{t}{t-1}\right)w(t) = 0, \quad \text{where} \quad w(t) = v'(t).$$

We get

$$w(t) = c_1 e^{-\int\left(\frac{2}{t} - \frac{t}{t-1}\right)dt}, \quad c_1 \text{ is arbitrary constant}$$

$$-\int\left(\frac{2}{t} - \frac{t}{t-1}\right)dt = -\int\left(\frac{2}{t} - 1 - \frac{1}{t-1}\right)dt = t + \log\left(\frac{t-1}{t^2}\right)$$

and so

$$w(t) = c_1 e^t\left(\frac{t-1}{t^2}\right) = c_1\left(\frac{e^t}{t} - \frac{e^t}{t^2}\right) = v'(t).$$

Finally, we get

$$v(t) = c_1\frac{e^t}{t} + c_2, \quad c_1, \ c_2 \text{ are arbitrary constants,}$$

which gives

$$\begin{aligned} y_2(t) &= tv = t\left(c_1\frac{e^t}{t} + c_2\right) \\ &= c_1 e^t \text{ (new solution)} + c_2 t \text{ (old solution)}, \quad t \in (1, \infty). \end{aligned}$$

Therefore, the general solution of (66) is given by

$$y(t) = c_1 t + c_2 e^t, \quad t \in (1, \infty). \tag{67}$$

The proof is done. □

Remark 2.23 ***(Interesting observation.)*** *By (67) we see that any solution of (66) is actually defined for all $t \in (-\infty, \infty)$, i.e. it can be defined across $t = 1$.*

2.1.2 Reduction method for nonhomogeneous equations.

Remark 2.24 *(**Be careful.**) Throughout this section, we will focus on equation (68), which has **leading coefficient** 1 for $y''(t)$.*

The **reduction method** can also be used to solve the **nonhomogeneous** equation

$$y''(t) + p(t) y'(t) + q(t) y(t) = g(t), \quad t \in I. \tag{68}$$

Assume that $y_1(t)$ is a solution to the **homogeneous** equation $y''(t)+p(t)y'(t)+q(t)y(t) = 0$ on I and $y_1(t) \neq 0$ on I. Let $y_2(t) = v(t) y_1(t)$ to get

$$y_1(t) v''(t) + [2y_1'(t) + p(t) y_1(t)] v'(t) = g(t) \quad \text{(compare with (62))}, \tag{69}$$

which is a first order linear ODE for $w(t) := v'(t)$. By solving (69), one can get $y_2(t)$ and $Y(t)$, where $y_2(t)$ is a solution of $y''(t)+py'+qy = 0$ and $Y(t)$ is a solution of $y''+py'+qy = g$. Now the general solution of (68) is given by (assume that $y_1(t)$ and $y_2(t)$ have **nonzero** Wronskian)

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \quad t \in I.$$

Example 2.25 *Use the above reduction method to solve the nonhomogeneous equation*

$$y''(t) + 4y'(t) + 4y(t) = e^{3t}, \quad t \in (-\infty, \infty). \tag{70}$$

<u>**Solution**</u>:

We know $y_1(t) = e^{-2t}$ is one solution of $y''(t) + 4y'(t) + 4y(t) = 0$. Let

$$y_2(t) = v(t) e^{-2t}$$

and plug it into the nonhomogeneous equation (70) to get

$$e^{-2t} v''(t) = e^{3t} \quad \text{(compare with (65))}.$$

We obtain

$$v(t) = \frac{1}{25} e^{5t} + c_1 t + c_2,$$

which gives

$$y_2(t) = \left(\frac{1}{25} e^{5t} + c_1 t + c_2 \right) e^{-2t} = \frac{1}{25} e^{3t} + \left(c_1 t e^{-2t} + c_2 e^{-2t} \right).$$

Note that $Y(t) = \frac{1}{25} e^{3t}$ is a particular solution of the nonhomogeneous equation (70) and $y_2(t) = te^{-2t}$ is another solution of the corresponding homogeneous equation. Thus the general solution of (70) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{1}{25} e^{3t}, \quad t \in (-\infty, \infty)$$

for arbitrary constants $c_1,\ c_2$.

Remark 2.26 *(**Be careful ...**) In case we know a particular solution $Y(t)$ to the equation $y''(t) + p(t) y'(t) + q(t) y(t) = g(t)$ on I. Then if we try to use reduction method using the solution $Y(t)$, **it will not work in general**. To see this, let $y_2(t) = v(t) Y(t)$ and plug it into the nonhomogeneous equation (68) to get (see (62) first)*

$$\begin{aligned} & y_2''(t) + p(t) y_2'(t) + q(t) y_2(t) \\ &= Y(t) v''(t) + [2Y'(t) + p(t) Y(t)] v'(t) + \left[\underbrace{Y''(t) + p(t) Y'(t) + q(t) Y(t)} \right] v(t) \qquad (71) \\ &= Y(t) v''(t) + [2Y'(t) + p(t) Y(t)] v'(t) + \underbrace{g(t) v(t)} = g(t). \end{aligned}$$

*Unfortunately for this situation, we are **not able** to solve $v(t)$ in general, because it **cannot be reduced to a first order equation**.*

2.2 Use Wronskian to solve a second order linear homogeneous equation (Wronskian method) (see p. 174, Exercise 32).

Remark 2.27 *(**Be careful.**) Throughout this section, we will focus on equation (72), which has **leading coefficient** 1 for $y''(t)$.*

Remark 2.28 *Unlike the reduction method, the **Wronskian method** in this section is valid only for **homogeneous equation. The Wronskian method is as good as the reduction method.***

Consider the second order linear homogeneous equation (note that the coefficient of $y''(t)$ is 1 in (72))

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \quad t \in I \tag{72}$$

and assume that **we know one solution** $y_1(t)$ of (72) on I and it satisfies $y_1(t) \neq 0$ **everywhere on** I.

Let $y_2(t)$ be another new solution ($y_2(t)$ is not a multiple of $y_1(t)$) of the above equation to be solved. The Wronskian $W(t)$ of $y_1(t),\ y_2(t)$ satisfies

$$W(t) = Ce^{-\int p(t)dt}, \quad C \neq 0 \text{ (since } y_2(t) \text{ is not a multiple of } y_1(t)\text{)}. \tag{73}$$

Therefore, we get

$$y_1(t)y_2'(t) - y_1'(t)y_2(t) = Ce^{-\int p(t)dt}, \quad C \neq 0, \quad t \in I. \tag{74}$$

This gives a first order **linear** equation for $y_2(t)$ and **since we assume** $y_1(t) \neq 0$ **on** I, **we can solve** $y_2(t)$ **on the whole interval** I (to see this, just divide the equation (74) by $y_1(t)$ on I).

If we solve $y_2(t)$ on I from equation (74), then $y_2(t)$ will be a solution of (72) on I. More precisely, we have:

Lemma 2.29 *(**Wronskian method of solving ODE (72).**) Let $y_1(t)$ be a solution of (72) on I satisfying $y_1(t) \neq 0$ everywhere on I. Let $C \neq 0$ be any given constant. If $y_2(t)$ is a solution of the linear equation (74) on I, then it is also a solution of (72) on I. Moreover, $y_2(t)$ is not a constant multiple of $y_1(t)$ on I and $\{y_1(t),\ y_2(t)\}$ forms a **fundamental set of solutions** to the ODE (72) on I.*

Remark 2.30 *By Lemma 2.29, one can obtain a new solution $y_2(t)$ different from $y_1(t)$ (by solving the linear equation (74) on I) and then obtain a **fundamental set of solutions** $\{y_1(t),\ y_2(t)\}$. Every solution $y(t)$ on I is a linear combination of $y_1(t)$ and $y_2(t)$ on I.*

Proof. We have

$$y_1(t)y_2'(t) - y_1'(t)y_2(t) = Ce^{-\int p(t)dt}, \quad C \neq 0, \quad t \in I \tag{75}$$

and by differentiation, we get

$$y_1(t)y_2''(t) - \underbrace{y_1''(t)}\,y_2(t) = (-p(t)) \cdot Ce^{-\int p(t)dt}, \quad \forall\, t \in I.$$

Since $y_1(t)$ is a solution of (72) on I, the above identity becomes

$$\begin{aligned} &y_1(t)y_2''(t) + \left[\underbrace{p(t)y_1'(t) + q(t)y_1(t)}\right] y_2(t) = (-p(t)) \cdot Ce^{-\int p(t)dt} \\ &= (-p(t)) \cdot [y_1(t)y_2'(t) - y_1'(t)y_2(t)], \end{aligned}$$

which can be simplified as

$$y_1(t)[y_2''(t) + p(t)y_2'(t) + q(t)y_2(t)] = 0, \quad \forall\, t \in I.$$

As we know that $y_1(t) \neq 0$ for all $t \in I$, we must have $y_2''(t) + p(t) y_2'(t) + q(t) y_2(t) = 0$ for all $t \in I$.

Finally, we see that $y_2(t)$ is not a constant multiple of $y_1(t)$ on I. If so, we will have $y_1(t) y_2'(t) - y_1'(t) y_2(t) \equiv 0$ on I, which gives a contradiction due to $C \neq 0$ in (75). Therefore, $\{y_1(t), \ y_2(t)\}$ forms a **fundamental set of solutions** to the ODE (72) on I. The proof is done. □

Remark 2.31 *If the constant C is 0 in equation (74), we will get $y_2(t) = \lambda y_1(t)$ for arbitrary constant λ, which is useless for us.*

Remark 2.32 *(**Important.**) Be careful that when you use the Wronskian method, make sure you rewrite the equation into the form $y'' + p(t) y' + q(t) y = 0$ first.*

Remark 2.33 *(**Important.**) Note that the Wronskian method is valid only for **homogeneous equations.** This is because the Abel's formula (73) for $W(t)$ is valid only for homogeneous equation (see Theorem 2.4).*

Example 2.34 *(**Example 3 in p. 172.**) We shall use reduction method and Wronskian method to solve the equation*

$$2t^2 y'' + 3ty' - y = 0, \quad t \in (0, \infty),$$

given that $y(t) = 1/t$ is a solution of it, which is not equal to 0 on $(0, \infty)$.

Remark 2.35 *This equation is, in fact, an Euler equation. So we know how to obtain its general solution. However, here we want to use different methods and see how they work.*

<u>**Solution**</u>:

1. Reduction method (rewriting equation to have leading coefficient 1 first):

We first rewrite the equation as

$$y'' + \frac{3}{2t} y' - \frac{1}{2t^2} y = 0, \quad t \in (0, \infty)$$

and let $y(t) = v(t) t^{-1}$ and by the formula

$$y_1(t) v''(t) + [2y_1'(t) + p(t) y_1(t)] v'(t) = 0, \quad y_1(t) = \frac{1}{t}$$

we get

$$\frac{1}{t} v''(t) + \left[2\left(-\frac{1}{t^2}\right) + \frac{3}{2t}\frac{1}{t} \right] v'(t) = 0, \quad t \in (0, \infty),$$

which gives

$$w' - \frac{1}{2t} w = 0, \quad w = v', \quad t \in (0, \infty)$$

and then

$$w(t) = v'(t) = Ce^{\int \frac{1}{2t} dt} = Ct^{1/2}, \quad t \in (0, \infty).$$

Hence $v(t) = \frac{2}{3} C t^{3/2} + k$ and

$$y(t) = v(t) t^{-1} = \frac{2}{3} C t^{1/2} + kt^{-1}.$$

Thus the general solutionis given by ($t^{1/2}$ and t^{-1} is a **fundamental set of solutions** on $(0, \infty)$)

$$y(t) = C_1 t^{1/2} + C_2 t^{-1}.$$

2. Wronskian method (rewriting equation to have leading coefficient 1 first):

We first rewrite the equation as

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0, \quad t \in (0, \infty).$$

Let $y_1(t) = t^{-1}$ and we want to find $y_2(t)$. By Abel's Theorem, the Wronskian $W(t)$ of $y_1(t)$ and $y_2(t)$ is given by

$$W(t) = y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t) = Ce^{-\int p(t)dt} = Ce^{-\int \frac{3}{2t}dt} = Ct^{-3/2}, \quad C \neq 0 \text{ is a const..}$$

Hence we get

$$\frac{1}{t}y_2'(t) + \frac{1}{t^2}y_2(t) = Ct^{-3/2},$$

i.e.,

$$y_2'(t) + \frac{1}{t}y_2(t) = Ct^{-1/2}, \quad t \in (0, \infty).$$

We obtain

$$y_2(t) = e^{-\int \frac{1}{t}dt}\left[\int \left(e^{\int \frac{1}{t}dt} Ct^{-1/2}\right) dt + \tilde{C}\right], \quad \tilde{C} \text{ is another const.}$$

$$= \frac{1}{t}\left(C\int t^{1/2}dt + \tilde{C}\right) = \frac{1}{t}\left(\frac{2}{3}Ct^{3/2} + \tilde{C}\right) = \frac{2}{3}Ct^{1/2} + \tilde{C}t^{-1}, \quad t \in (0, \infty).$$

Thus the general solution is given by

$$y(t) = C_1 t^{1/2} + C_2 t^{-1}, \quad t \in (0, \infty).$$

We get the same result as in the reduction method. □

Method of undetermined coefficients (this is Section 3.5 of the book). See p. 182, Table 3.5.1.

Remark 2.36 *For second order* ***nonhomogeneous*** *linear equation with* ***constant*** *coefficients, the "****method of undetermined coefficients****" provides you a way to "****guess the form****" of a* ***particular*** *solution when the nonhomogeneous term $g(t)$ is given by (77) below. Then we plug in the form into the equation to find a* ***correct*** *particular solution.*

In this section, we consider a second order **nonhomogeneous** linear equation with **constant** coefficients, given by

$$ay''(t) + by'(t) + cy(t) = g(t), \quad a,\ b,\ c \text{ are const.}, \quad a \neq 0, \quad t \in (-\infty, \infty), \tag{76}$$

where $g(t)$ has one of the following forms (**this is a necessary condition**)

$$P_n(t)\, e^{\lambda t}, \qquad P_n(t)\, e^{\alpha t}\cos\beta t, \qquad P_n(t)\, e^{\alpha t}\sin\beta t, \tag{77}$$

and $P_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1}t + a_n$, $a_0 \neq 0$, is a **polynomial** with degree n and $\lambda,\ \alpha,\ \beta \in \mathbb{R}$, $\beta > 0$ (the case $\lambda = 0$ and the case $\alpha = 0$ are allowed). In case $\lambda = 0$ and $\alpha = 0$, $P_n(t)\, e^{\lambda t} = P_n(t)$ is just a polynomial in t and $P_n(t)\, e^{\alpha t}\cos\beta t$ becomes $P_n(t)\cos\beta t$, etc.

We know that the general solution $y(t)$ of (76) is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t), \quad t \in (-\infty, \infty), \tag{78}$$

where $y_p(t)$ is a **particular solution** of the nonhomogeneous equation (76) and $y_1(t)$, $y_2(t)$ are solutions of $ay''(t) + by'(t) + cy(t) = 0$, determined by the roots of the characteristic equation $ar^2 + br + c = 0$. Since we know how to find $y_1(t)$, $y_2(t)$, it suffices to find a particular solution $y_p(t)$ of (76).

The **"method of undetermined coefficients"** says that we can try a **particular solution** of the **form** given by **Table 3.5.1 in p. 182** of the book and then plug in the form into the **nonhomogeneous equation** (76) to **determine the coefficients**. After that, one can find a particular solution $y_p(t)$.

Remark 2.37 *Explain Table 3.5.1 in p. 182 ....*

Remark 2.38 ***(Important.)*** *The function $g(t)$ in equation (76) must have the form in (77); otherwise, the "method of undetermined coefficients"* ***does not work****.*

Motivation of the undetermined coefficients method.

We can use first order nonhomogeneous equation as a motivation to understand the undetermined coefficients method for second order nonhomogeneous equation.

Motivation using the equation $y'(t) - \lambda y(t) = a_0 e^{\alpha t}$, where $\lambda,\ a_0 \neq 0,\ \alpha$ are constants. Consider the simple equation

$$y'(t) - \lambda y(t) = a_0 e^{\alpha t}, \quad a_0,\ \lambda,\ \alpha \text{ are constants, } a_0 \neq 0. \tag{79}$$

The **characteristic equation of the homogeneous equation** $y'(t) - \lambda y(t) = 0$ is $r - \lambda = 0$, which has root $r = \lambda$ and so the general solution of $y'(t) - \lambda y(t) = 0$ is given by $y(t) = Ce^{\lambda t}$ for arbitrary constant C. To find the general solution of (79), it suffices to find a **particular solution** $y_p(t)$.

Case 1: If $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), then a function of the form

$$y_p(t) = \underbrace{A_0 e^{\alpha t}}, \quad A_0 \text{ is a constant to be determined,} \tag{80}$$

can be a particular solution of the ODE (79). If we plug in the above $y_p(t)$ into the equation (79), we get

$$y_p'(t) - \lambda y_p(t) = A_0(\alpha - \lambda) e^{\alpha t} = a_0 e^{\alpha t}, \quad \text{where } \alpha - \lambda \neq 0. \tag{81}$$

Therefore, if choose $A_0 = a_0/(\alpha - \lambda)$ in (80), $y_p(t)$ will be a particular solution of the ODE (79).

Case 2: If $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), then identity (81) becomes $0 = a_0 e^{\alpha t}$, which is **impossible to hold** and the function (80) **cannot be** a particular solution. Instead, if we try $y_p(t)$ to have the form

$$y_p(t) = \underbrace{t \cdot A_0 e^{\alpha t}}, \quad B_0 \text{ is a constant to be determined,} \tag{82}$$

and plug in the above $y_p(t)$ in (82) into the equation (79), we get

$$y_p'(t) - \lambda y_p(t) = A_0 e^{\alpha t} + tA_0(\alpha - \lambda) e^{\alpha t} = A_0 e^{\alpha t} = a_0 e^{\alpha t}, \quad \text{where} \quad \alpha = \lambda. \tag{83}$$

Therefore, if choose $A_0 = a_0$ in (82), $y_p(t)$ will be a particular solution of the ODE (79).

We can summarize the following:

Lemma 2.39 *Consider the first order nonhomogeneous linear equation*

$$y'(t) - \lambda y(t) = P_0(t) e^{\alpha t}, \quad \lambda,\ \alpha \text{ are constants,} \tag{84}$$

where $P_0(t)$ *is a nonzero polynomial with degree* 0 *(i.e.* $P_0(t) = a_0$ *is a* ***nonzero constant****). Then a* ***particular solution*** $y_p(t)$ *of the ODE (84) has the* ***form***

$$y_p(t) = \begin{cases} Q_0(t)\, e^{\alpha t}, & \alpha \neq \lambda, \\ t \cdot Q_0(t)\, e^{\alpha t}, & \alpha = \lambda, \end{cases} \tag{85}$$

where $Q_0(t)$ *is also a nonzero polynomial with degree* 0 *(the same degree as* $P_0(t)$*).*

Motivation using the equation $y'(t) - \lambda y(t) = (a_0 + b_0 t)\, e^{\alpha t}$**, where** $\lambda,\ a_0,\ b_0 \neq 0,\ \alpha$ **are constants.** We look at the equation

$$y'(t) - \lambda y(t) = (a_0 + b_0 t) e^{\alpha t}, \quad a_0,\ b_0,\ \lambda,\ \alpha \text{ are constants, } b_0 \neq 0. \tag{86}$$

Case 1: If $\alpha \neq \lambda$ (i.e. α is **not a root** of the characteristic equation $r - \lambda = 0$), we first try

$$y_p(t) = (A_0 + B_0 t)\, e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0, \tag{87}$$

and see if it works. Plug it into equation (86) to get

$$y_p'(t) - \lambda y_p(t) = B_0 e^{\alpha t} + \alpha (A_0 + B_0 t)\, e^{\alpha t} - \lambda (A_0 + B_0 t)\, e^{\alpha t} = (a_0 + b_0 t)\, e^{\alpha t},$$

which is same as

$$B_0 + (\alpha - \lambda)(A_0 + B_0 t) = a_0 + b_0 t, \quad \text{where } \alpha - \lambda \neq 0. \tag{88}$$

and if we choose $A_0,\ B_0$ satisfying

$$\begin{cases} B_0 + (\alpha - \lambda) A_0 = a_0, \\ (\alpha - \lambda) B_0 = b_0, \quad \alpha - \lambda \neq 0, \end{cases}$$

i.e.

$$A_0 = \frac{a_0}{\alpha - \lambda} - \frac{b_0}{(\alpha - \lambda)^2}, \qquad B_0 = \frac{b_0}{\alpha - \lambda}, \qquad \alpha \neq \lambda, \tag{89}$$

then $y_p(t)$ in (87) will be a **particular solution** of the ODE (86).

Remark 2.40 *In case we have* $a_0 = 0$*, i.e. equation with the form*

$$y'(t) - \lambda y(t) = b_0 t e^{\alpha t},$$

we still need to choose $y_p(t) = (A_0 + B_0 t)\, e^{\alpha t}$ *to obtain the correct particular solution. In such a case, we have*

$$y_p(t) = \left(-\frac{b_0}{(\alpha - \lambda)^2} + \frac{b_0}{\alpha - \lambda} t \right) e^{\alpha t}.$$

Case 2: If $\alpha = \lambda$ (i.e. α is **a root** of the characteristic equation $r - \lambda = 0$), then the identity (88) becomes $B_0 = a_0 + b_0 t$, which is **impossible to hold**. Therefore you need to modify your choice of $y_p(t)$ in (87). A natural further choice is (increase the order of the coefficient polynomial):

$$y_p(t) = \left(A_0 + B_0 t + C_0 t^2\right) e^{\alpha t} \quad \text{for some constants} \quad A_0,\ B_0,\ C_0.$$

However, note that $A_0 e^{\alpha t}$ is a solution of the homogeneous equation $y'(t) - \lambda y(t) = 0$ (note that now $\alpha = \lambda$), there is **no need** to include it. Hence we choose

$$y_p(t) = \left(B_0 t + C_0 t^2\right) e^{\alpha t} = t\,(B_0 + C_0 t)\, e^{\alpha t}$$

and for consistency of notations, we write it as

$$y_p(t) = t \cdot (A_0 + B_0 t) e^{\alpha t} \quad \text{for some constants} \quad A_0, \ B_0. \tag{90}$$

If we plug the above $y_p(t)$ into (86), we get

$$y_p'(t) - \lambda y_p(t) = (A_0 + B_0 t) e^{\alpha t} + t B_0 e^{\alpha t} = (a_0 + b_0 t) e^{\alpha t}, \quad \text{where} \quad \alpha = \lambda$$

and conclude

$$A_0 = a_0, \quad B_0 = \frac{b_0}{2}.$$

Thus when $\alpha = \lambda$, the function

$$y_p(t) = t \cdot \left(a_0 + \frac{b_0}{2} t \right) e^{\alpha t}, \quad t \in (-\infty, \infty)$$

will be a **particular solution** of the equation (86).

Again, we can summarize the following:

Lemma 2.41 *(**Motivation of the undetermined coefficients method via first-order equation.**) Consider the first order nonhomogeneous linear equation ODE*

$$y'(t) - \lambda y(t) = P_1(t) e^{\alpha t}, \quad \lambda, \ \alpha \text{ are constants,} \tag{91}$$

*where $P_1(t)$ is a nonzero polynomial with degree 1 (i.e. $P_1(t) = a_0 + b_0 t, \ b_0 \neq 0$). Then a **particular solution** $y_p(t)$ of the ODE (84) has the **form***

$$y_p(t) = \begin{cases} Q_1(t) e^{\alpha t}, & \alpha \neq \lambda, \\ t \cdot Q_1(t) e^{\alpha t}, & \alpha = \lambda, \end{cases} \tag{92}$$

where $Q_1(t)$ is also a nonzero polynomial with degree 1 (the same degree as $P_1(t)$).

From Lemma **??**, you can understand the undetermined coefficients method in Table 3.5.1 in p. 182 of the book.

Remark 2.42 *State the rule in Table 3.5.1 in p. 182 of the textbook here again.*

Explain the following examples ....

Example 2.43 $y'' + 3y = 4e^{-5t}, \quad y_p(t) = \frac{1}{7} e^{-5t}.$

Example 2.44 $y'' - 3y' - 4y = 2e^{-t}, \quad y_p(t) = -\frac{2}{5} t e^{-t}.$

Example 2.45 $y'' + 2y = \sin 3t, \quad y_p(t) = -\frac{1}{7} \sin 3t$ *(since there is no first order term y', the solution $y_p(t)$ is also of the form $\sin 3t$).*

Example 2.46 $y'' + 9y = \sin 3t, \quad y_p(t) = -\frac{1}{6} t \cos 3t.$

Example 2.47 $y'' - 3y = t^2, \quad y_p(t) = -\frac{1}{3} t^2 - \frac{2}{9}.$

Example 2.48 $y'' - 3y' = t + t^2, \quad y_p(t) = t \left(-\frac{1}{9} t^2 - \frac{5}{18} t - \frac{5}{27} \right).$

Example 2.49 *Do Example 3 in p. 179.*

Example 2.50 *Find general solution of the equation*

$$y'' + 2y' + y = te^{-t}.$$

Solution:

By the rule for $y_p(t)$, it has the form

$$y_p(t) = t^s (At + B) e^{-t} = \left(At^3 + Bt^2\right) e^{-t}, \quad \text{where } s = 2.$$

Plugging it into equation to get

$$\left\{ \begin{array}{l} \left[(6At + 2B) e^{-t} - 2\left(3At^2 + 2Bt\right) e^{-t} + \left(At^3 + Bt^2\right) e^{-t}\right] \\ +2\left[\left(3At^2 + 2Bt\right) e^{-t} - \left(At^3 + Bt^2\right) e^{-t}\right] + \left(At^3 + Bt^2\right) e^{-t} \end{array} \right. = te^{-t}.$$

Hence, after simplification, we need to solve $6At + 2B = t$, which gives

$$A = \frac{1}{6}, \quad B = 0.$$

Thus $y_p(t) = \frac{1}{6}t^3e^{-t}$ is a particular solution of the equation. The general solution is

$$y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{6}t^3e^{-t}, \quad t \in (-\infty, \infty).$$

□

Remark 2.51 *If an equation has the form*

$$ay'' + by' + cy = f(t) + g(t), \tag{93}$$

where $f(t)$ and $g(t)$ both have the form in the above case 1 or case 2 (say $f(t) = t^2e^{5t}$ and $g(t) = \left(t^3 + 2t^2 - 6t - 5\right) e^{-t} \cos 7t$), then use the undetermined coefficients to find $y_p(t)$ for the equation

$$ay'' + by' + cy = f(t)$$

and then use the same method to find $\tilde{y}_p(t)$ for the equation

$$ay'' + by' + cy = g(t).$$

Then the general solution of (93) is given by

$$x(t) = y_p(t) + \tilde{y}_p(t) + c_1y_1(t) + c_2y_2(t),$$

where $c_1x_1(t) + c_2x_2(t)$ is the general solution of the corresponding homogeneous equation.

Example 2.52 *Find the correct form of a particular solution of the equation*

$$y'' - 4y' + 4y = 3t^2e^{2t} + 2t\sin t - 8e^t \cos 2t.$$

Solution:

The correct form is

$$y_p(t) = \underbrace{t^2\left(At^2 + Bt + C\right) e^{2t}} + \underbrace{(Dt + E) \cos t + (Ft + G) \sin t} + \underbrace{Ke^t \cos 2t + Le^t \sin 2t},$$

where A, ..., L are constant coefficients to be determined. □

Example 2.53 *(**This is Exercise 30 in p. 185 with one extra term.**) Find the general solution of the equation*

$$y'' + \lambda^2 y = \sum_{m=1}^{N} (a_m \sin m\pi t + b_m \cos m\pi t) \quad t \in (-\infty, \infty), \tag{94}$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1,\ 2,\ ...,\ N$.

Remark 2.54 *Note that for each fixed m, the particular solution $y_m(t)$ for $a_m \sin m\pi t$ and the particular solution $\tilde{y}_m(t)$ for $b_m \cos m\pi t$ have the same form. Therefore, $a_m \sin m\pi t + b_m \cos m\pi t$ can be viewed as just one nonhomogeneous function.*

Solution:

The two roots of the characteristic polynomial $r^2 + \lambda^2 = 0$ are $r = \pm\lambda i$, where $\lambda \neq m\pi$ for any $m = 1,\ ...,\ N$. Hence **for each** fixed $m = 1,\ ...,\ N$, according to the method, we can try a particular solution $y_m(t)$ of the form

$$y_m(t) = A_m \sin m\pi t + B_m \cos m\pi t, \tag{95}$$

which is for the equation

$$y'' + \lambda^2 y = a_m \sin m\pi t + b_m \cos m\pi t. \tag{96}$$

We plug the above $y_m(t)$ into equation (96) to get

$$\left(\lambda^2 - m^2\pi^2\right) A_m \sin m\pi t + \left(\lambda^2 - m^2\pi^2\right) B_m \cos m\pi t = a_m \sin m\pi t + b_m \cos m\pi t$$

and obtain

$$A_m = \frac{a_m}{\lambda^2 - m^2\pi^2}, \qquad B_m = \frac{b_m}{\lambda^2 - m^2\pi^2}, \qquad m = 1,\ ...,\ N.$$

Hence, the general solution of the equation is given by (add all $y_m(t)$ together):

$$y(t) = c_1 \sin \lambda t + c_2 \cos \lambda t + \sum_{m=1}^{N} \left(\frac{a_m}{\lambda^2 - m^2\pi^2}\right) \sin m\pi t + \left(\frac{b_m}{\lambda^2 - m^2\pi^2}\right) \cos m\pi t.$$

The proof is done. □

Variation of parameters method (this is Section 3.6 of the book) for nonhomogeneous linear equations with variable coefficients.

Remark 2.55 *(**Be careful.**) Throughout this section, we will focus on equation (97), which has leading coefficient 1 for $y''(t)$.*

In this section we focus on a **nonhomogeneous linear equation with variable coefficients** (which has **leading coefficient** 1), given by

$$y'' + p(t)\, y' + q(t)\, y = g(t), \quad t \in I, \tag{97}$$

where $p(t),\ q(t),\ g(t)$ are given continuous function on some interval $I \subseteq \mathbb{R}$ and here the function $g(t)$ can be an **arbitrary.**

Assume we are given a **fundamental set of solutions** $y_1(t)$ and $y_2(t)$ for the corresponding homogeneous equation $y'' + p(t)\, y' + q(t)\, y = 0$ on I. To solve (97), we try a solution of the form:

$$y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t), \quad t \in I \tag{98}$$

and look for suitable $u_1(t)$ and $u_2(t)$. We will solve a **first-order system of ODE** for $u_1(t)$ and $u_2(t)$.

Remark 2.56 *(**Useful observation.**) One can view (98) as a generalization of the **reduction method** because if we only try $y(t) = u_1(t) y_1(t)$, it is exactly the reduction method.*

Remark 2.57 *If an equation has the form*

$$P(t) y''(t) + Q(t) y'(t) + R(t) y(t) = G(t), \quad t \in I, \quad P(t) \neq 0 \text{ on } I, \tag{99}$$

*then **you should divide the whole equation by** $P(t)$ **first and then apply the method below.***

We need to impose suitable conditions on $u_1(t)$ and $u_2(t)$ so that the above $y(t)$ is a solution of (97). We first note that

$$y'(t) = \underbrace{[u_1'(t) y_1(t) + u_2'(t) y_2(t)]} + [u_1(t) y_1'(t) + u_2(t) y_2'(t)]$$

and **impose the first condition**

$$\underbrace{u_1'(t) y_1(t) + u_2'(t) y_2(t)} = 0, \quad t \in I. \tag{100}$$

Remark 2.58 *If we impose the condition on the term $u_1(t) y_1'(t) + u_2(t) y_2'(t)$, then in $y''(t)$ we will encounter $u_1''(t)$ and $u_2''(t)$. With this, the method will not work at all.*

Then, under the assumption of (100), $y'(t)$ becomes

$$y'(t) = u_1(t) y_1'(t) + u_2(t) y_2'(t), \quad t \in I$$

and so

$$y''(t) = \underbrace{[u_1'(t) y_1'(t) + u_2'(t) y_2'(t)]} + [u_1(t) y_1''(t) + u_2(t) y_2''(t)], \quad t \in I.$$

Then we **impose the second condition** as

$$\underbrace{u_1'(t) y_1'(t) + u_2'(t) y_2'(t)} = g(t). \tag{101}$$

Under the assumption of (100) and (101), we conclude

$$\begin{cases} y'(t) = u_1(t) y_1'(t) + u_2(t) y_2'(t), \\ y''(t) = g(t) + u_1(t) y_1''(t) + u_2(t) y_2''(t), \end{cases} \tag{102}$$

and so

$$\begin{aligned} & y''(t) + p(t) y'(t) + q(t) y(t) \\ & = [g(t) + u_1(t) y_1''(t) + u_2(t) y_2''(t)] + p(t)[u_1(t) y_1'(t) + u_2(t) y_2'(t)] + q(t)[u_1(t) y_1(t) + u_2(t) y_2(t)] \\ & = g(t) + u_1(t) \left[\underbrace{y_1''(t) + p(t) y_1'(t) + q(t) y_1(t)} \right] + u_2(t) \left[\underbrace{y_2''(t) + p(t) y_2'(t) + q(t) y_2(t)} \right] \\ & = g(t) + u_1(t) \cdot 0 + u_2(t) \cdot 0 = g(t), \quad t \in I, \end{aligned}$$

which says that $y(t)$ is indeed a solution of the nonhomogeneous equation (97).

It remains to claim that (100) and (101) can be satisfied. For this purpose, we need to solve the following **first-order system of ODE** for $u_1(t)$ and $u_2(t)$:

$$\begin{cases} u_1'(t) y_1(t) + u_2'(t) y_2(t) = 0 \\ u_1'(t) y_1'(t) + u_2'(t) y_2'(t) = g(t) \end{cases} \tag{103}$$

and get

$$u_1'(t) = \frac{\begin{vmatrix} 0 & y_2(t) \\ g(t) & y_2'(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = -\frac{y_2(t)\,g(t)}{W(t)}, \qquad u_2'(t) = \frac{\begin{vmatrix} y_1(t) & 0 \\ y_1'(t) & g(t) \end{vmatrix}}{\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}} = \frac{y_1(t)\,g(t)}{W(t)},$$

where $W(t) = W(y_1, y_2)(t)$ is the **Wronskian** of $y_1(t)$ and $y_2(t)$ on I.

The above gives

$$u_1(t) = -\int \frac{y_2(t)\,g(t)}{W(t)}dt + c_1, \qquad u_2(t) = \int \frac{y_1(t)\,g(t)}{W(t)}dt + c_2, \tag{104}$$

and the general solution of (97) is given by (98), i.e.

$$\begin{aligned} y(t) &= \left(-\int \frac{y_2(t)\,g(t)}{W(t)}dt + c_1\right) y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(t)}dt + c_2\right) y_2(t) \\ &= c_1 y_1(t) + c_2 y_2(t) + y_p(t), \end{aligned}$$

where

$$y_p(t) = \underbrace{-\left(\int \frac{y_2(t)\,g(t)}{W(t)}dt\right) y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(t)}dt\right) y_2(t)} \tag{105}$$

is a **particular solution** of (97). The above method is called "**variation of parameters**" method. It is a powerful method.

Remark 2.59 ***(Important.)*** *If the equation (97) has initial conditions $y(t_0) = y_0,\ y'(t_0) = z_0,\ t_0 \in I$, then there are two ways to find the unique solution $y(t)$. (1). If you know $y_p(t)$* ***explicitly****, use the formula $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$ to find $c_1,\ c_2$. (2). If you* ***do not know*** *$y_p(t)$ explicitly, you can use definite integrals to write the general solution $y(t)$ as*

$$y(t) = \begin{cases} c_1 y_1(t) + c_2 y_2(t) \\ + \left[-\left(\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(s)}ds\right) y_1(t) + \left(\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(s)}ds\right) y_2(t)\right], \quad t \in I \end{cases} \tag{106}$$

and then require $c_1,\ c_2$ to satisfy the following

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) = z_0. \end{cases}$$

This is due to the fact that the ***particular solution***

$$y_p(t) = -\left(\int_{t_0}^{t} \frac{y_2(s)\,g(s)}{W(s)}ds\right) y_1(t) + \left(\int_{t_0}^{t} \frac{y_1(s)\,g(s)}{W(s)}ds\right) y_2(t), \quad t \in I \tag{107}$$

satisfies

$$y(t_0) = y'(t_0) = 0. \tag{108}$$

To see this, we clearly have $y_p(t_0) = 0$. As for $y_p'(t_0) = 0$, we note that

$$y_p'(t_0) = -\left(\frac{y_2(t_0)\,g(t_0)}{W(t_0)}\right) y_1(t_0) + 0 \cdot y_1'(t_0) + \left(\frac{y_1(t_0)\,g(t_0)}{W(t_0)}\right) y_2(t_0) + 0 \cdot y_2'(t_0) = 0. \tag{109}$$

One can also write $y_p(t)$ in (107) as

$$y_p(t) = \int_{t_0}^{t} \frac{y_1(s)\,y_2(t) - y_1(t)\,y_2(s)}{y_1(s)\,y_2'(s) - y_1'(s)\,y_2(s)} g(s)\,ds, \quad t \in I. \tag{110}$$

The above formula appears in p. 191 of the textbook.

Example 2.60 ***(This is Example 1 in p. 186.)*** *Solve the equation*

$$y''(t) + 4y(t) = 3\csc t, \quad 0 < t < \pi, \quad \csc t = \frac{1}{\sin t}. \tag{111}$$

Solution:

We choose $y_1(t) = \cos 2t,\ y_2(t) = \sin 2t$, and get their Wronskian $W(t) = 2$. By (105), we have

$$y_p(t) = -\left(\int \frac{\sin 2t \cdot 3\csc t}{2} dt\right)\cos 2t + \left(\int \frac{\cos 2t \cdot 3\csc t}{2} dt\right)\sin 2t,$$

where

$$-\int \frac{\sin 2t \cdot 3\csc t}{2} dt = -\int \frac{2\sin t\cos t \cdot 3\frac{1}{\sin t}}{2} dt = -3\int \cos t dt = -3\sin t$$

and

$$\begin{aligned}\int \frac{\cos 2t \cdot 3\csc t}{2} dt &= \int \frac{\left(1 - 2\sin^2 t\right)\cdot 3\frac{1}{\sin t}}{2} dt \\ &= \frac{3}{2}\int \csc t dt - 3\int \sin t dt = \frac{3}{2}\log|\csc t - \cot t| + 3\cos t.\end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(t) &= c_1\cos 2t + c_2\sin 2t + y_p(t) \\ &= \left\{\begin{array}{l} c_1\cos 2t + c_2\sin 2t \\ -3\sin t\cdot\cos 2t + \left(\frac{3}{2}\log|\csc t - \cot t| + 3\cos t\right)\sin 2t, \quad t\in(0,\pi), \end{array}\right.\end{aligned}$$

where $c_1,\ c_2$ are arbitrary constants. □

Example 2.61 ***(This is Exercise 5 in p. 190.)*** *Solve the equation*

$$y''(t) + y(t) = 2\tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}. \tag{112}$$

Remark 2.62 *Note that one cannot use the undetermined coefficients method to solve (112).*

Solution:

Since we know two independent solutions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ of $y''(t) + y(t) = 0$, we can use variation of parameters method. We first compute

$$W(t) = W(y_1, y_2)(t) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

and by (105) we know that the general solution of (112) is given by

$$y(t) = c_1\cos t + c_2\sin t + \left(-\int \frac{y_2(t)\cdot 2\tan t}{W(t)} dt\right) y_1(t) + \left(\int \frac{y_1(t)\cdot 2\tan t}{W(t)} dt\right) y_2(t),$$

where

$$\begin{aligned} &-\int \frac{y_2(t)\cdot 2\tan t}{W(t)} dt \\ &= -2\int \sin t\cdot\tan t dt = -2\int \frac{\left(1-\cos^2 t\right)}{\cos} dt = -2\int \sec t dt + 2\int \cos t dt \end{aligned}$$

and

$$\int \frac{y_1(t) \cdot 2\tan t}{W(t)} dt = 2\int \cos t \cdot \tan t dt = 2\int \sin t dt.$$

The general solution is

$$\begin{aligned} y(t) &= c_1 \cos t + c_2 \sin t + \left(-2\int \sec t dt + 2\int \cos t dt\right)\cos t + \left(2\int \sin t dt\right)\sin t \\ &= c_1 \cos t + c_2 \sin t + \left(-2\int \sec t dt\right)\cos t \\ &= c_1 \cos t + c_2 \sin t + \left(-2\log\left|\sec t + \tan t\right|\right)\cos t, \end{aligned} \tag{113}$$

where $c_1,\ c_2$ are arbitrary constants. □

Example 2.63 ***(This is Exercise 10 in p. 190.)*** *Solve the equation*

$$y''(t) - 2y'(t) + y(t) = \frac{e^t}{1+t^2}, \quad t \in (-\infty, \infty).$$

<u>**Solution**</u>:

Since we know two independent solutions $y_1(t) = e^t$ and $y_2(t) = te^t$ of $y''(t) - 2y'(t) + y(t) = 0$, we can use variation of parameters method. We first compute

$$W(t) = W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t}$$

and so

$$\begin{aligned} y(t) &= \left(-\int \frac{te^t \cdot \frac{e^t}{1+t^2}}{e^{2t}} dt + c_1\right)e^t + \left(\int \frac{e^t \cdot \frac{e^t}{1+t^2}}{e^{2t}} dt + c_2\right)te^t \\ &= \left(-\int \frac{t}{1+t^2} dt + c_1\right)e^t + \left(\int \frac{1}{1+t^2} dt + c_2\right)te^t \\ &= \left(-\frac{1}{2}\log\left(1+t^2\right) + c_1\right)e^t + \left(\tan^{-1} t + c_2\right)te^t, \end{aligned}$$

which is the general solution. □

Example 2.64 *Find the general solution of the equation*

$$ty''(t) - (1+t)y'(t) + y(t) = t^2 e^{2t}, \quad t \in (0, \infty), \tag{114}$$

given that $y_1(t) = 1+t$ and $y_2(t) = e^t$ is a pair of ***fundamental set of solutions*** *for the corresponding homogeneous equation.*

<u>**Solution**</u>:

To apply the variation of parameters method, **we need to rewrite the equation to have leading coefficient of $y''(t)$ equal to** 1. We have

$$y''(t) - \left(\frac{1+t}{t}\right)y'(t) + \frac{1}{t}y(t) = te^{2t}, \quad t \in (0, \infty),$$

and obtain $g(t) = te^{2t}$. By the variation of parameters method, we have

$$y_p(t) = \left(-\int \frac{y_2(t)\,g(t)}{W(y_1, y_2)(t)} dt\right)y_1(t) + \left(\int \frac{y_1(t)\,g(t)}{W(y_1, y_2)(t)} dt\right)y_2(t), \quad t \in (0, \infty),$$

where

$$W(y_1, y_2)(t) = \begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = te^t.$$

Hence

$$\begin{aligned} y_p(t) &= \left(-\int \frac{e^t \cdot te^{2t}}{te^t} dt\right)(1+t) + \left(\int \frac{(1+t)\cdot te^{2t}}{te^t} dt\right) e^t \\ &= \left(-\int e^{2t} dt\right)(1+t) + \left(\int (1+t) e^t dt\right) e^t \\ &= \left(-\frac{1}{2} e^{2t}\right)(1+t) + \left(te^t\right) e^t = \frac{1}{2}(t-1) e^{2t}. \end{aligned}$$

and conclude the general solution for equation (114):

$$y(t) = C_1 (1+t) + C_2 e^t + \frac{1}{2}(t-1) e^{2t}, \quad t \in (0, \infty).$$

□

An interesting equation from "mechanics of vibrations".

Consider the equation

$$y''(t) + y(t) = g(t), \quad g(t) \text{ is continuous on } I$$

with initial condition

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \quad t_0 \in I.$$

This equation appears frequently in **mechanics of vibrations (spring vibrations without friction, with or without outer force)**. If we choose $y_1(t) = \cos t$, $y_2(t) = \sin t$, then we have $W(t) = W(y_1, y_2)(t) = 1$ and by Remark 2.59 the **particular solution** $y_p(t)$ (it satisfies $y_p(t_0) = y_p'(t_0) = 0$) in (107) is given by

$$\begin{aligned} y_p(t) &= -\left(\int_{t_0}^t \frac{y_2(s) g(s)}{W(s)} ds\right) y_1(t) + \left(\int_{t_0}^t \frac{y_1(s) g(s)}{W(s)} ds\right) y_2(t), \quad W(s) \equiv 1 \\ &= -(\cos t) \int_{t_0}^t g(s) \sin s ds + (\sin t) \int_{t_0}^t g(s) \cos s ds \\ &= \int_{t_0}^t g(s)(\sin t \cos s - \cos t \sin s) ds = \underbrace{\int_{t_0}^t g(s) \sin(t-s) ds}, \quad t \in I. \end{aligned} \tag{115}$$

The general solution of the homogeneous equation $y''(t) + y(t) = 0$ satisfying the initial condition $y(t_0) = y_0$, $y'(t_0) = z_0$, is given by

$$c_1 y_1(t) + c_2 y_2(t) = c_1 \cos t + c_2 \sin t, \quad t \in I$$

and we need to find c_1, c_2 satisfying

$$\begin{cases} c_1 \cos t_0 + c_2 \sin t_0 = y_0 \\ -c_1 \sin t_0 + c_2 \cos t_0 = z_0, \end{cases}$$

which gives

$$c_1 = y_0 \cos t_0 - z_0 \sin t_0, \quad c_2 = y_0 \sin t_0 + z_0 \cos t_0,$$

and then

$$\begin{aligned}
&c_1 y_1(t) + c_2 y_2(t) \\
&= (y_0 \cos t_0 - z_0 \sin t_0) \cos t + (y_0 \sin t_0 + z_0 \cos t_0) \sin t \\
&= y_0 \cos(t - t_0) + z_0 \sin(t - t_0) . \qquad (116)
\end{aligned}$$

Therefore the solution satisfying the initial condition is given by the **nice solution formula**:

$$y(t) = \underbrace{y_0 \cos(t - t_0) + z_0 \sin(t - t_0)} + \underbrace{\int_{t_0}^{t} g(s) \sin(t - s)\, ds}, \quad t \in I. \qquad (117)$$

Now assume that $I = (-\infty, \infty)$ and $g(t)$ is a 2π**-periodic function** defined on $(-\infty, \infty)$ ($g(t)$ usually comes from the **external force** acting on the mechanical system, say **string vibration**). **The particular solution** $y_p(t)$ **in (115) may not be** 2π**-periodic in general** (but the homogeneous part $y_0 \cos(t - t_0) + z_0 \sin(t - t_0)$ is clearly 2π-periodic). Note that we have

$$\begin{aligned}
&y_p(t + 2\pi) - y_p(t) \\
&= \int_{t_0}^{t+2\pi} g(s) \sin(t + 2\pi - s)\, ds - \int_{t_0}^{t} g(s) \sin(t - s)\, ds = \int_{t}^{t+2\pi} g(s) \sin(t - s)\, ds \\
&= -(\cos t) \underbrace{\int_{t}^{t+2\pi} g(s) \sin s ds} + (\sin t) \underbrace{\int_{t}^{t+2\pi} g(s) \cos s ds} \\
&= -(\cos t) \int_{0}^{2\pi} g(s) \sin s ds + (\sin t) \int_{0}^{2\pi} g(s) \cos s ds \\
&= \left\langle (-\cos t, \sin t), \left(\int_{0}^{2\pi} g(s) \sin s ds, \int_{0}^{2\pi} g(s) \cos s ds \right) \right\rangle
\end{aligned}$$

and so we have $y_p(t + 2\pi) = y_p(t)$ for all $t \in (-\infty, \infty)$ **if and only if** the 2π-periodic function $g(s)$ satisfies

$$\int_{0}^{2\pi} g(s) \sin s ds = \int_{0}^{2\pi} g(s) \cos s ds = 0. \qquad (118)$$

If we take $g(t) = \cos t$ ((118) is not satisfied), then one can check that $y_p(t) = \frac{1}{2} t \sin t$ is a particular solution (with $y_p(0) = y_p'(0) = 0$) of the equation

$$y''(t) + y(t) = \cos t,$$

but it is **not** 2π-periodic even if $g(t) = \cos t$ is 2π-periodic. In fact, one can see that $y_p(t)$ in (115) is 2π-periodic **if and only if** $g(t)$ is 2π-periodic and satisfies (118), for example, say $g(t) = \cos 2t$.

Nonhomogeneous Euler equation.

One can combine the variation of parameters method and change of variables to solve a **nonhomogeneous Euler equation**, given by

$$t^2 y''(t) + \alpha t y'(t) + \beta y(t) = f(t), \quad t \in (0, \infty), \quad \alpha,\ \beta \text{ constants}, \qquad (119)$$

where $f(t)$ can be any **arbitrary** continuous function defined on $t \in (0, \infty)$. By the change of variables $x = \log t,\ x \in (-\infty, \infty)$, the above equation becomes

$$\frac{d^2 \tilde{y}}{dx^2} + (\alpha - 1) \frac{d\tilde{y}}{dx} + \beta \tilde{y}(x) = F(x), \quad x \in (-\infty, \infty), \qquad (120)$$

where $\tilde{y}(x) = y(e^x)$ and $F(x) = f(e^x)$. We can know a pair of **fundamental set of solutions** $\{\tilde{y}_1(x), \tilde{y}_2(x)\}$ for $\tilde{y}''(x) + (\alpha - 1)\tilde{y}'(x) + \beta\tilde{y}(x) = 0$ and then use the **variation of parameters method** to find the general solution $\tilde{y}(x)$ of (120) and then change back to get $y(t)$. It will be the general solution of (119). On the other hand, if $F(x)$ in (120) has the forms appeared in the **undetermined coefficients method**, then you can use that method to find the general solution $\tilde{y}(x)$ of (120).

Remark 2.65 *In case the Euler equation has the form*

$$At^2y''(t) + Bty'(t) + Cy(t) = f(t), \quad t \in (0, \infty), \quad A \neq 0, \ B, \ C \text{ are constants,}$$

then equation (120) becomes

$$A\frac{d^2\tilde{y}}{dx^2} + (B - A)\frac{d\tilde{y}}{dx} + C\tilde{y}(x) = F(x). \tag{121}$$

Summary of solution methods for second order linear equations.

Remark 2.66 *Read the following summary by yourself.*

This is a summary for solving a nonhomogeneous second order linear equation.

Case 1: $ay'' + by' + cy = g(t)$, $t \in I$, **where** a, b, c **are constants with** $a \neq 0$ **and** $g(t)$ **is a continuous nonzero function on** I.

In this case you can easily find two independent solutions $y_1(t)$ and $y_2(t)$ of $ay'' + by' + cy = 0$.

(1). In case $g(t)$ is of the form $P_n(t)e^{\lambda t}$, $P_n(t)e^{\alpha t}\cos\beta t$, $P_n(t)e^{\alpha t}\sin\beta t$, where $P_n(t)$ is a polynomial with degree n and λ, α, $\beta \in \mathbb{R}$ with $\beta > 0$, use the **method of undetermined coefficients** (the easiest way).

(2). In case $g(t)$ is not of the form in (1), first **rewrite the equation as**

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = \frac{g(t)}{a}, \quad a \neq 0. \tag{122}$$

Then you can use **decomposition method** (if the characteristic polynomial $ar^2 + br + c = 0$ has **two real roots)**, or **reduction method**, or **variation of parameters method** (but not Wronskian method since $g(t)$ is a nonzero function). **Variation of parameters method seems to be the best one** because we know two independent solutions $y_1(t)$, $y_2(t)$ of the equation $ay'' + by' + cy = 0$.

Case 2: $y'' + p(t)y' + q(t)y = g(t)$, $t \in I$, **where** $p(t)$, $q(t)$, $g(t)$ **are continuous functions on** I.

Remark 2.67 ***(Be careful.)*** *Here the coefficient of* $y''(t)$ *is* 1. *If the equation is of the form*

$$P(t)y''(t) + Q(t)y'(t) + R(t)y(t) = G(t), \quad t \in I, \quad P(t) \neq 0 \text{ on } I, \tag{123}$$

then you should divide the whole equation by $P(t)$ *first and then apply the following summary.*

Here we **assume that we are given one nonzero solution** $y_1(t)$ **of the homogeneous equation** $y'' + p(t)y' + q(t)y = 0$ **on** I.

(1). In case $g(t) \equiv 0$ on I and we know one solution $y_1(t)$ of the **homogeneous** equation $y'' + p(t)y' + q(t)y = 0$, use **reduction method** or **Wronskian method**.

(2). In case $g(t)$ is **a nonzero** function on I, use **reduction method.**

(3). In case $g(t)$ is a **nonzero** functionon I and we know two independent solutions $y_1(t)$, $y_2(t)$ (**fundamental set of solutions**) of $y'' + p(t) y' + q(t) y = 0$ on I, use **variation of parameters method**.

(4). In case equation is of the form then

$$y'' + \frac{\alpha}{t} y' + \frac{\beta}{t^2} y = g(t), \quad t \in (0, \infty), \quad \alpha, \ \beta \text{ are const.},$$

then use "**nonhomogeneous Euler equation**" method (see Section 2.2) to solve it.

Chapter 4: Higher order linear equations.

Section 4.1: General theory of n-th order linear equations.

Consider the n-**th order linear equation (with leading coefficient** 1**)**

$$y^{(n)}(t) + p_1(t) y^{(n-1)}(t) + \cdots + p_{n-1}(t) y'(t) + p_n(t) y(t) = g(t), \quad t \in I \tag{124}$$

where $p_1(t)$, ..., $p_n(t)$, $g(t)$ are given and continuous on I. Since equation (124) is linear, by ODE theory, any solution $y(t)$ to equation (124) is defined **on the whole interval** I.

When the **initial conditions**

$$y(t_0) = y_0, \quad y'(t_0) = z_0, \quad \cdots, \quad y^{(n-1)}(t_0) = \gamma_0 \tag{125}$$

are given, we have the **existence and uniqueness theorem** (see Theorem 4.1.1 in p. 222 of the book). Moreover, the unique solution $y(t)$ is defined on the whole interval $t \in I$.

We now state the following:

Lemma 2.68 ***(Abel's formula for homogeneous equation.)*** *Let $y_1(t)$, ..., $y_n(t)$ be solutions of the homogeneous equation*

$$y^{(n)}(t) + p_1(t) y^{(n-1)}(t) + \cdots + p_{n-1}(t) y'(t) + p_n(t) y(t) = 0, \quad t \in I. \tag{126}$$

Define its ***Wronskian*** *$W(y_1, \ldots, y_n)(t)$ as in the book (for simplicity, denote it as $W(t)$) (see p. 223 of the book). Then we have*

$$W(t) = Ce^{-\int p_1(t) dt}, \quad t \in I, \tag{127}$$

for some constant C. Therefore, either $W(t) \equiv 0$ or $W(t) \neq 0$ for all $t \in I$.

Remark 2.69 ***(Important.)*** *Be careful that the leading coefficient in equation (126) is* 1 *and also that the equation is homogeneous.*

Remark 2.70 *If $W(t) \neq 0$ on I, $\{y_1(t), \ldots, y_n(t)\}$ is called a* ***fundamental set of solutions*** *of equation (126) on I.*

Proof. We have

$$W(t) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}, \quad t \in I. \tag{128}$$

Compute

$$\frac{dW}{dt}(t) = \det(M(t)),$$

where $M(t)$ is $n \times n$ matrix whose first $n-1$ rows are **unchanged** (i.e. the same as the first $n-1$ rows of (128)) and whose n-th row is $\left(y_1^{(n)}(t), \dots, y_n^{(n)}(t)\right)$, where we know that

$$y_1^{(n)}(t) = -\left[p_1(t)\, y_1^{(n-1)}(t) + \cdots + p_{n-1}(t)\, y_1'(t) + p_n(t)\, y_1(t)\right]$$

and the same for $y_2^{(n)}(t), \dots, y_n^{(n)}(t)$. By the **expansion property of determinant**, we have

$$\frac{dW}{dt}(t) = \det(M(t)) = -p_1(t)\, W(t), \quad \forall\, t \in I. \tag{129}$$

The proof is done. □

Remark 2.71 *To understand the identity (129), we can look at the case $n = 3$ and verify it. Note that*

$$\begin{aligned}\frac{dW}{dt}(t) &= \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}(t) \\ &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1y_1'' - p_2y_1' - p_3y_1 & -p_1y_2'' - p_2y_2' - p_3y_2 & -p_1y_3'' - p_2y_3' - p_3y_3 \end{vmatrix}(t).\end{aligned}$$

We know that if we multiply the first row by $p_3(t)$ and add it onto the third row, the determinant is ***unchanged****. Similarly, if we multiply the second row by $p_2(t)$ and add it onto the third row, the determinant is unchanged. By this, we have*

$$\frac{dW}{dt}(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1y_1'' & -p_1y_2'' & -p_1y_3'' \end{vmatrix}(t) = -p_1(t)\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}(t) = -p_1(t)\, W(t)$$

for all $t \in I$. Hence (129) is verified.

The most important result for equation (124) is the following, which is similar to Theorem 2.10:

Theorem 2.72 ***(This is Theorem 4.1.2 in p. 223.)*** *Let $y_1(t), \dots, y_n(t)$, $t \in I$, be a set of solutions of the following linear homogeneous equation*

$$y^{(n)}(t) + p_1(t)\, y^{(n-1)}(t) + \cdots + p_{n-1}(t)\, y'(t) + p_n(t)\, y(t) = 0, \quad t \in I. \tag{130}$$

Then the family of solutions

$$y(t) = c_1y_1(t) + \cdots + c_ny_n(t), \quad t \in I, \tag{131}$$

with ***arbitrary*** *constants $c_1, \dots, c_n$, will* ***generate all possible solutions*** *of (130) on I* ***if and only if***

$$W(y_1, \dots, y_n)(t_0) \neq 0 \quad \text{for some} \quad t_0 \in I \tag{132}$$

(hence $W(y_1, \dots, y_n)(t) \neq 0$ for all $t \in I$).

Remark 2.73 *In the above theorem, we call*

$$y(t) = c_1y_1(t) + \cdots + c_ny_n(t), \quad t \in I, \tag{133}$$

where $c_1, \dots, c_n$ are arbitrary constants, the ***general solution*** *of equation (130).*
We call $\{y_1(t), \dots, y_n(t)\}$ a ***fundamental set of solutions*** *of the ODE (130) on I.*

Proof. The idea of proof is similar to the previous case for second order linear equations (we need to use the **existence and uniqueness** property for equation (130) with initial conditions (125)). We omit it. $\square$

Corollary 2.74 *Let $y_1(t)$, ..., $y_n(t)$ be from the above theorem such that $W(y_1, \ldots, y_n)(t_0) \neq 0$ for some $t_0 \in I$. Also let $y_p(t)$ be a solution to the nonhomogeneous equation (124) on I. Then the* ***general solution*** *of equation (124) is given by*

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + y_p(t), \quad t \in I \tag{134}$$

for arbitrary constants c_1, ..., c_n.

Proof. We omit this. $\square$

Section 4.2: Homogeneous equations with constant coefficients.

Remark 2.75 *Just follow textbook for this section. See p. 229 for "real and unequal roots" and p. 230-232 for "complex and repeated roots".*

In this section, we look at an n-th order linear homogeneous equation with **constant coefficients**, given by

$$L[y] := a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1} y^{(1)}(t) + a_n y(t) = 0, \quad a_0 \neq 0, \quad t \in (-\infty, \infty), \tag{135}$$

where a_0, ..., a_n are constants with $a_0 \neq 0$. similar to the previous situation, the polynomial equation

$$p_n(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0 \tag{136}$$

is called the **characteristic equation** of the ODE. If we plug the function $y(t) = e^{rt}$ (r is a number, which can be real or complex) into (135), we get

$$L\left[e^{rt}\right] = p_n(r) e^{rt} = 0. \tag{137}$$

Therefore, if r is a **real** root of the polynomial equation $p_n(r) = 0$, the function $y(t) = e^{rt}$ is a **real solution** of (135).

To go further, we recall that there is a **decomposition property** for polynomials with **real coefficients**, which says:

Lemma 2.76 *Let $p_n(r)$ be a polynomial with degree $n \in \mathbb{N}$ given by (136). Then it has a unique (up to permutation of factors) decomposition of the form*

$$p_n(r) = a_0 (r - \lambda_1)(r - \lambda_2) \cdots (r - \lambda_m)(r - z_1)(r - \bar{z}_1) \cdots (r - z_k)(r - \bar{z}_k), \tag{138}$$

*where λ_1, ..., λ_m are real numbers (**may be repeated**) and $z_1 = \alpha_1 + i\beta_1$ ($\beta_1 > 0$), ..., $z_k = \alpha_k + i\beta_k$ ($\beta_k > 0$) are complex numbers (**may be repeated**). Without involving complex numbers, we can also decompose $p_n(r)$ in real numbers as*

$$p_n(r) = a_0 (r - \lambda_1)(r - \lambda_2) \cdots (r - \lambda_m) \left[r^2 - 2\alpha_1 r + \left(\alpha_1^2 + \beta_1^2\right)\right] \cdots \left[r^2 - 2\alpha_k r + \left(\alpha_k^2 + \beta_k^2\right)\right]. \tag{139}$$

Let D be the differentiation operator with $Dy = y'(t)$, $D^2 y = y''(t)$, $(D^2 - 4D + 5)y = y''(t) - 4y'(t) + 5y(t)$, etc. We can express the equation (135) as

$$\left(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n\right) y = 0$$

and for simplicity, also denote it as

$$p_n(D)\, y = 0, \quad \text{where } p_n(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

For linear equation with constant coefficients, an important property is:

Lemma 2.77 *Let $y(t)$ be any n-times differentiable function on I and*

$$p_n(r) = a_0(r-\lambda_1)(r-\lambda_2)\cdots(r-\lambda_m)\left[r^2-2\alpha_1 r+\left(\alpha_1^2+\beta_1^2\right)\right]\cdots\left[r^2-2\alpha_k r+\left(\alpha_k^2+\beta_k^2\right)\right]. \quad (140)$$

We have the identity on I :

$$p_n(D)\,y = \left[\begin{array}{l} a_0(D-\lambda_1)(D-\lambda_2)\cdots(D-\lambda_m)\left(\underbrace{D^2-2\alpha_1 D+\left(\alpha_1^2+\beta_1^2\right)}\right)\cdots \\ \cdots\ \underbrace{\left(D^2-2\alpha_k D+\left(\alpha_k^2+\beta_k^2\right)\right)} \end{array}\right] y, \quad (141)$$

where the right hand side of (141) means we apply the operator $[D^2-2\alpha_k D+(\alpha_k^2+\beta_k^2)]$ on y first and then followed by the operator $\left[D^2-2\alpha_{k-1}D+\left(\alpha_{k-1}^2+\beta_{k-1}^2\right)\right]$ and repeat the same process until we use up all the differentiation operator. Moreover, the identity (141) still holds if we ***change the order of differentiation operators*** *on the right hand side of (141).*

Proof. One can use mathematical induction to prove Lemma 2.77. We omit it. □

Example 2.78 *Let*

$$p_4(D) = D^4+4D^3+10D^2+12D+9 = \left(D^2+2D+3\right)^2.$$

We have

$$p_4(D)\,y = y'''' + 4y''' + 10y'' + 12y' + 9y.$$

On the other hand, we also have

$$\begin{aligned} &\left[\left(D^2+2D+3\right)\left(D^2+2D+3\right)\right]y = \left[\left(D^2+2D+3\right)\right](y''+2y'+3y) \\ &= (y''+2y'+3y)'' + 2(y''+2y'+3y)' + 3(y''+2y'+3y) \\ &= (y''''+2y'''+3y'') + 2(y'''+2y''+3y') + 3(y''+2y'+3y) \\ &= y''''+4y'''+10y''+12y'+9y. \end{aligned}$$

The identity (141) is verified.

Lemma 2.79 *Let $r \neq 0$ be a real number and $m,\ n \in \mathbb{N}\bigcup\{0\}$. We have the following identities*

$$\begin{cases} (D-r)^n\left(t^m e^{rt}\right) = \frac{m!}{(m-n)!}t^{m-n}e^{rt}, & n \le m, \\ (D-r)^m\left(t^m e^{rt}\right) = m!e^{rt}, & n = m, \\ (D-r)^{m+1}\left(t^m e^{rt}\right) = (D-r)\left[(D-r)^m\left(t^m e^{rt}\right)\right] = (D-r)\left(m!e^{rt}\right) = 0, & n = m+1 > m. \end{cases} \quad (142)$$

In particular, we have

$$(D-r)^n\left(t^m e^{rt}\right) = 0 \quad \text{if} \quad m \in \{0,\ 1,\ 2,\ \cdots,\ n-1\}, \quad n \in \mathbb{N}. \quad (143)$$

Therefore for $n \in \mathbb{N}$ the ODE $(D-r)^n y(t) = 0$ has the following n solutions:

$$e^{rt},\ te^{rt},\ t^2e^{rt},\ t^3e^{rt},\ \ldots,\ t^{n-1}e^{rt}, \quad t \in (-\infty,\infty). \quad (144)$$

By Wronskian theory (you can check that their Wronskian $W(t) \neq 0$ on I). They form a ***fundamental set of solutions*** *of the ODE $(D-r)^n y(t) = 0$ on $t \in (-\infty,\infty)$.*

Proof. This is a straightforward verification. Prove it by yourself. □

Remark 2.80 *We check for the case $n = 3$ that the Wronskian of the solutions in (144) is indeed nonzero. We have*

$$W(t) = \begin{vmatrix} e^{rt} & te^{rt} & t^2e^{rt} \\ re^{rt} & e^{rt} + rte^{rt} & 2te^{rt} + rt^2e^{rt} \\ r^2e^{rt} & 2re^{rt} + r^2te^{rt} & 2e^{rt} + 4rte^{rt} + r^2t^2e^{rt} \end{vmatrix}$$
$$= e^{rt} \cdot e^{rt} \cdot e^{rt} \begin{vmatrix} 1 & t & t^2 \\ r & 1 + rt & 2t + rt^2 \\ r^2 & 2r + r^2t & 2 + 4rt + r^2t^2 \end{vmatrix}, \quad r \neq 0,$$

and we can multiply the first row by $-r$ and add onto the second row, and multiply the first row by $-r^2$ and add onto the third row, to get

$$W(t) = e^{3rt} \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 2r & 2 + 4rt \end{vmatrix}, \quad r \neq 0.$$

Finally we multiply the second row by $-2r$ and add onto the third row, to get

$$W(t) = e^{3rt} \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2e^{3rt} \neq 0, \ \forall \ t \in (-\infty, \infty). \tag{145}$$

To go on, we recall that for any constants $A, \ B, \ \alpha, \ \beta > 0$, we have

$$\left(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\right) \left(Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t\right) = 0, \quad \forall \ t \in (-\infty, \infty). \tag{146}$$

Similar to Lemma 2.79, we have:

Lemma 2.81 *Let $\alpha, \ \beta \in \mathbb{R}$ with $\beta > 0$ and $m \in \mathbb{N}$. For any constants $A, \ B$, we have*

$$\left(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\right) \left[t^m \cdot \left(Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t\right)\right]$$
$$= \begin{cases} D^2 \left[t^m \cdot (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)\right] \\ -2\alpha D \left[t^m \cdot (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)\right] + (\alpha^2 + \beta^2) \left[t^m \cdot (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)\right] \end{cases}$$
$$= \begin{cases} m(m-1)\, t^{m-2} \cdot (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t) + 2\left[mt^{m-1} \cdot D(Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)\right] \\ +\underline{t^m \cdot D^2 (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)} \\ -2\alpha m t^{m-1} \cdot (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t) - \underline{t^m \cdot 2\alpha D (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)} \\ +\underline{t^m \cdot (\alpha^2 + \beta^2)(Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)}, \end{cases}$$

where by (146) we have

$$\begin{cases} \underline{t^m \cdot D^2 (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)} - \underline{t^m \cdot 2\alpha D (Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)} \\ +\underline{t^m \cdot (\alpha^2 + \beta^2)(Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t)} \end{cases}$$
$$= t^m \left(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\right) \left(Ae^{\alpha t} \cos \beta t + Be^{\alpha t} \sin \beta t\right) = 0, \quad \forall \ t \in (-\infty, \infty).$$

Hence we conclude

$$\Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)\left[t^m \cdot \left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right)\right]$$
$$= \begin{cases} m\,(m-1)\,t^{m-2} \cdot (Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t) + 2\left[mt^{m-1} \cdot D\,(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t)\right] \\ -2\alpha m t^{m-1} \cdot (Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t) \end{cases}$$
$$= \left(C_1 t^{m-1} + C_2 t^{m-2}\right)\left(C_3 e^{\alpha t}\cos\beta t + C_4 e^{\alpha t}\sin\beta t\right), \quad m \in \mathbb{N}, \quad t \in (-\infty, \infty), \tag{147}$$

where C_1, ..., C_4 *are constants depending on* α, β, m, A, B.

Proof. This is a straightforward verification. Prove it by yourself.

Corollary 2.82 *In particular, for* $m = 1$ *in Lemma 2.81, we have*

$$\Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)\left[t \cdot \left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right)\right]$$
$$= (2D - 2\alpha)\left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right)$$
$$= C_3 e^{\alpha t}\cos\beta t + C_4 e^{\alpha t}\sin\beta t, \quad \forall\, t \in (-\infty, \infty) \tag{148}$$

for some constants C_3, C_4 *depending on* α, β, A, B. *Finally, by (146) and (147), we have*

$$\Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)^2\left[t\left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right)\right] = 0, \quad \forall\, t \in (-\infty, \infty).$$

Therefore, the four functions

$$e^{\alpha t}\cos\beta t, \quad e^{\alpha t}\sin\beta t, \quad te^{\alpha t}\cos\beta t, \quad te^{\alpha t}\sin\beta t, \quad t \in (-\infty, \infty)$$

are solutions of the ***4-th order*** *ODE on* $(-\infty, \infty)$:

$$\Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)^2 y\,(t) = 0, \quad t \in (-\infty, \infty).$$

They form a ***fundamental set of solutions*** *of the ODE on* $t \in (-\infty, \infty)$.

By (148) in Corollary 2.82, we have:

Lemma 2.83 *Let* α, $\beta \in \mathbb{R}$ *with* $\beta > 0$ *and* $m \in \mathbb{N}$. *For any constants* A, B, *we have*

$$\Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)^{m+1}\left[t^m \cdot \left(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t\right)\right] = 0. \tag{149}$$

In particular, we also have

$$\begin{cases} \Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)^{m+1}(Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t) = 0, \\ \Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)^{m+1}\left[t \cdot (Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t)\right] = 0, \\ \cdots \\ \Big(\underbrace{D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)}\Big)^{m+1}\left[t^m \cdot (Ae^{\alpha t}\cos\beta t + Be^{\alpha t}\sin\beta t)\right] = 0, \end{cases}$$

and the $2(m+1)$ *functions*

$$e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t,\ te^{\alpha t}\cos\beta t,\ te^{\alpha t}\sin\beta t,\ \cdots,\ t^m e^{\alpha t}\cos\beta t,\ t^m e^{\alpha t}\sin\beta t,\quad t\in(-\infty,\infty)$$

are solutions of the ODE

$$\Big(\underbrace{D^2-2\alpha D+\left(\alpha^2+\beta^2\right)}\Big)^{m+1} y=0,\quad t\in(-\infty,\infty).$$

They form a ***fundamental set of solutions*** *of the ODE on* $t\in(-\infty,\infty)$.

With the help of Lemma 2.79 and Lemma 2.83, you can understand the examples in the textbook.

Example 2.84 *Find the general solution of the equation (135) where* $n=14$ *and the 14 roots of the characteristic equation* $p_n(r)=0$ *are given by*

$$0,\ 0,\ -4,\ 7,\ -5,\ -5,\ -5,\ -5,\ 3+2i,\ 3+2i,\ 3+2i,\ 3-2i,\ 3-2i,\ 3-2i.$$

The answer is

$$y(t)=\begin{cases} c_1+c_2t+c_3e^{-4t}+c_4e^{7t}+\left(c_5+c_6t+c_7t^2+c_8t^3\right)e^{-5t} \\ +\left(c_9+c_{10}t+c_{11}t^2\right)e^{3t}\cos 2t+\left(c_{12}+c_{13}t+c_{14}t^2\right)e^{3t}\sin 2t,\end{cases}$$

where $t\in(-\infty,\infty)$.

Example 2.85 *(**This is Example 3 in p. 232.**) Consider the ODE*

$$y''''+2y''+y=0.$$

Its characteristic equation is $p(r)=r^4+2r^2+1=0$, *which has four roots*

$$r=i,\ i,\ -i,\ -i.$$

Therefore, its general solution is given by

$$y(t)=c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t,\quad t\in(-\infty,\infty).$$

Example 2.86 *(**This is Example 4 in p. 233.**) Let* $p(r)=r^4+1$. *Then we can write it as (completing the square)*

$$p(r)=r^4+2r^2+1-2r^2=\left(r^2+1\right)^2-\left(\sqrt{2}r\right)^2,$$

which gives the decomposition

$$p(r)=\left(r^2+\sqrt{2}r+1\right)\left(r^2-\sqrt{2}r+1\right).$$

Hence the four ***different*** *roots of the equation* $r^4+1=0$ *are*

$$\frac{-\sqrt{2}\pm\sqrt{2}i}{2},\qquad \frac{\sqrt{2}\pm\sqrt{2}i}{2}.$$

The general solution of the ODE $y''''+y=0$ *is given by*

$$y(t)=\begin{cases} C_1e^{(\sqrt{2}/2)t}\cos\left(\frac{\sqrt{2}}{2}\right)t+C_2e^{(\sqrt{2}/2)t}\sin\left(\frac{\sqrt{2}}{2}\right)t \\ +C_3e^{(-\sqrt{2}/2)t}\cos\left(\frac{\sqrt{2}}{2}\right)t+C_4e^{(-\sqrt{2}/2)t}\sin\left(\frac{\sqrt{2}}{2}\right)t,\quad t\in(-\infty,\infty).\end{cases}$$

The above is the second part (Part II) of the notes

To be continued in the third part (Part III) on 2024-11-21